

COMPUTABILITY OF SEMICOMPUTABLE MANIFOLDS IN COMPUTABLE TOPOLOGICAL SPACES

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ABSTRACT. We study computable topological spaces and semicomputable and computable sets in these spaces. In particular, we investigate conditions under which semicomputable sets are computable. We prove that a semicomputable compact manifold M is computable if its boundary ∂M is computable. We also show how this result combined with certain construction which compactifies a semicomputable set leads to the conclusion that some noncompact semicomputable manifolds in computable metric spaces are computable.

1. INTRODUCTION

A real number is computable if it can be effectively approximated by a rational number with arbitrary precision [22]. A tuple $(x_1, \dots, x_n) \in \mathbb{R}^n$ is computable if x_1, \dots, x_n are computable numbers. A compact subset of \mathbb{R}^n is computable if it can be effectively approximated by a finite set of points with rational coordinates with arbitrary precision [3]. Each nonempty computable subset of \mathbb{R}^n contains computable points, moreover they are dense in it.

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a computable function (in the sense of [19, 24]) such that the set $f^{-1}(\{0\})$ is compact. Does $f^{-1}(\{0\})$ have to be a computable set?

It is known that there exists a computable function $f : \mathbb{R} \rightarrow \mathbb{R}$ which has zero-points and all of them lie in $[0, 1]$, but none of them is computable [20]. So $f^{-1}(\{0\})$ is a nonempty compact set which contains no computable point. In particular $f^{-1}(\{0\})$ is not computable, in fact we might say it is “far away from being computable”.

Hence for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f^{-1}(\{0\})$ is a compact set the implication

$$(1) \quad f \text{ computable} \Rightarrow f^{-1}(\{0\}) \text{ computable}$$

does not hold in general. The question is are there any additional assumptions under which (1) holds. It turns out that such assumptions exist and that certain topological properties of the set $f^{-1}(\{0\})$ play an important role in this sense.

But before explaining what are these topological properties, we will give another view to implication (1).

A compact subset S of \mathbb{R}^n is semicomputable if we can effectively enumerate all rational open sets which cover S . It turns out that a compact subset S of \mathbb{R}^n is semicomputable if and only if $S = f^{-1}(\{0\})$ for some computable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Therefore closely related to the question under which conditions (1) holds is the question under which conditions for $S \subseteq \mathbb{R}^n$ the following implication holds:

$$(2) \quad S \text{ semicomputable} \Rightarrow S \text{ computable.}$$

That (2) does not hold in general we conclude from the fact that (1) does not hold in general. However (2) does hold under some topological conditions on S .

In order to see what is the role of topology in view of (2), let us first observe the simple case when S is a line segment in \mathbb{R} . In [17] it is given an example of a number $\gamma \in \mathbb{R}$ such that $[\gamma, 1]$ is a semicomputable, but not a computable subset of \mathbb{R} . On the other hand, if a and b are computable numbers such that $a < b$, then $[a, b]$ is a computable set. So (2) does not hold in general if S is a line segment in \mathbb{R} , but it does hold under additional assumption that the endpoints of S are computable.

The line segments and the arcs are the same in \mathbb{R} , but in higher-dimensional Euclidean spaces the arcs are much more general than the line segments. In view of the previous fact the following question arises: does (2) holds if S is an arc in \mathbb{R}^n with computable endpoints?

The answer to this question is not obvious. That the answer is affirmative follows from the more general result of Miller [17]: every semicomputable topological sphere in \mathbb{R}^n is computable and every semicomputable cell in \mathbb{R}^n with computable boundary sphere is computable. Miller's pioneer work regarding conditions under which (2) holds shows that topology has an important role in view of these conditions.

That these results hold in a larger class of computable metric spaces were shown in [9]. The more general result was later proved in [10]: (2) holds if S is a compact manifold with computable boundary (see also [13]).

Topological properties can force a semicomputable set S to be computable not just when S is locally Euclidean. Chainable and circularly chainable continua are generalizations of arcs and topological circles and it is proved that (2) holds if S is a continuum chainable from a to b , where a and b are computable points, or S is a circularly chainable continuum which is not chainable [8, 12]. Certain results when the complement of S is disconnected can be found in [8, 11].

The notions of semicomputable and computable set can be generalized to non-compact sets and it turns out that (2) does not hold in general if S is a (noncompact) 1-manifold with computable boundary [4]. However, it is proved that (2) holds if S is a 1-manifold with computable boundary under additional condition that S has finitely many connected components. Certain conditions under which (2) holds if S is the graph of a function can be found in [1].

On the other hand, Kihara constructed in [15], as the answer to a question in [16], an example of a nonempty semicomputable compact set in the plane which is simply connected (in fact, it is contractible) and which does not contain any computable point. There also exists a semicomputable set of a positive measure without a computable point [21].

In Euclidean space a set is semicomputable if and only if it is co-computably enumerable. That a set $S \subseteq \mathbb{R}^n$ is co-computably enumerable (co-c.e.) means that its complement $\mathbb{R}^n \setminus S$ can be effectively covered by open balls. For example the famous Mandelbrot set is co-c.e. (see [7]).

In this paper we put the investigation of conditions under which (2) holds into the more general ambient space: computable topological space. The notion of a computable topological space is not new, for example see [26, 25]. We will use the notion of a computable topological space which corresponds to the notion of a SCT₂ space from [25] and we will investigate some of its aspects. We will see

how to each computable metric space can be naturally associated a computable topological space and how the notions of a semicomputable and a computable set can be easily extend to computable topological spaces.

The central part of this paper will be the proof of the main result, i.e. the proof of the fact that (2) holds in any computable topological space if S is a compact manifold with computable boundary. This will be a generalization of the result from [10]. Although we will rely on certain ideas from [10], the main challenge will be to adopt ideas and techniques from [10], which depend on the metric d in a computable metric space (X, d, α) , to an ambient in which we do not have any metric. For example, the notion that a set S is computable up to a set T , which means that for each $k \in \mathbb{N}$ we can effectively find finitely many points x_0, \dots, x_n such that each point of S is 2^{-k} -close to some x_i and each x_i is 2^{-k} -close to some point of T , was essential in [10] and it is not obvious how to transfer it in a nonmetric setting. Another example is the notion of the formal diameter of a set in a computable metric space which is a computable analogue of the diameter of a set in a metric space and which clearly does not make sense in a (computable) topological space.

The generalization of the result for manifolds from [10] to computable topological spaces does not only show that a metric in this context is not really important, but it also provides a possible tool for dealing with the problem of computability of a semicomputable noncompact set S in a computable metric space (X, d, α) . Namely, using a construction similar to the one-point compactification, we can assign to (X, d, α) a computable topological space T in such a way that, under this construction, S maps to a compact set S' in T and such that the computability of S' in T implies the computability of S in (X, d, α) . We will see how this gives that a semicomputable set in a computable metric space homeomorphic to \mathbb{R}^n (for some n) must be computable.

It should be mentioned that the uniform version of the result from [10] does not hold in general: there exists a sequence (S_i) of topological circles in \mathbb{R}^2 such that S_i is uniformly semi-computable, but not uniformly computable (Example 7 in [8]).

Here is how the paper is organized. In Section 2 we state some basic definitions and facts. In Section 3 we study the notion of a computable topological space and in Section 4 we examine effective separations of compact sets in computable topological spaces. In Section 5 we introduce the notion of local computable enumerability of a set as a preparation for Sections 6 and 7 in which we prove our main result: a semicomputable manifold in a computable topological space is computable if its boundary is semicomputable. In Section 8 we reduce the problem of computability of *noncompact* semicomputable sets in a computable metric space to the problem of computability of *compact* semicomputable sets in a computable topological space.

2. COMPUTABLE METRIC SPACES AND PRELIMINARIES

In this section we give some basic facts about computable metric spaces and some other preliminary facts. See [19, 24, 22, 23, 3, 2, 8].

2.1. Computable functions $\mathbb{N}^k \rightarrow \mathbb{R}$. Let $k \in \mathbb{N}$, $k \geq 1$. A function $f : \mathbb{N}^k \rightarrow \mathbb{Q}$ is said to be **computable** if there exists computable (i.e. recursive) functions

$f_0, f_1, f_2 : \mathbb{N}^k \rightarrow \mathbb{N}$ such that

$$f(x) = (-1)^{f_0(x)} \frac{f_1(x)}{f_2(x) + 1}$$

for each $x \in \mathbb{N}^k$. A function $f : \mathbb{N}^k \rightarrow \mathbb{R}$ is said to be **computable** if there exists a computable function $F : \mathbb{N}^{k+1} \rightarrow \mathbb{Q}$ such that

$$|f(x) - F(x, i)| < 2^{-i}$$

for each $x \in \mathbb{N}^k$ and each $i \in \mathbb{N}$. Of course, a function $\mathbb{N}^k \rightarrow \mathbb{R}^n$ or $\mathbb{N}^k \rightarrow \mathbb{Q}^n$, where $n \in \mathbb{N}$, $n \geq 1$, will be called **computable** if its component functions are computable.

Some elementary properties of computable functions $\mathbb{N}^k \rightarrow \mathbb{R}$ are stated in the following proposition.

- Proposition 2.1.** (i) *If $f, g : \mathbb{N}^k \rightarrow \mathbb{R}$ are computable, then $f + g, f - g, f \cdot g : \mathbb{N}^k \rightarrow \mathbb{R}$ are computable.*
(ii) *If $f : \mathbb{N}^k \rightarrow \mathbb{R}$ and $F : \mathbb{N}^{k+1} \rightarrow \mathbb{R}$ are functions such that F is computable and $|f(x) - F(x, i)| < 2^{-i}$ for each $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$, then f is computable.*
(iii) *If $f, g : \mathbb{N}^k \rightarrow \mathbb{R}$ are computable functions, then the set $\{x \in \mathbb{N}^k \mid f(x) > g(x)\}$ is computably enumerable.* \square

2.2. Computable metric spaces. A triple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space and $\alpha = (\alpha_i)$ is a sequence whose range is dense in (X, d) and such that the function $\mathbb{N}^2 \rightarrow \mathbb{R}$,

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable (see [1, 2, 23, 14]). For example, if $n \geq 1$ and d is the Euclidean metric on \mathbb{R}^n , then for any computable function $\alpha : \mathbb{N} \rightarrow \mathbb{R}^n$ whose range is dense in \mathbb{R}^n we have that $(\mathbb{R}^n, d, \alpha)$ is a computable metric space. (Such a function α certainly exists: we can take a computable surjection $\alpha : \mathbb{N} \rightarrow \mathbb{Q}^n$.)

Let us recall the notion of the Hausdorff distance. If (X, d) is a metric space and S and T nonempty compact sets in this space, we define their **Hausdorff distance** $d_H(S, T)$ by

$$d_H(S, T) = \inf\{\varepsilon > 0 \mid S \approx_\varepsilon T\},$$

where $S \approx_\varepsilon T$ means that for each $x \in S$ there exists $y \in T$ such that $d(x, y) < \varepsilon$ and for each $y \in T$ there exists $x \in S$ such that $d(y, x) < \varepsilon$.

Let (X, d, α) be a computable metric space and $x \in X$. Then for each $k \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $d(x, \alpha_i) < 2^{-k}$. We say that x is a **computable point** in (X, d, α) if there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(3) \quad d(x, \alpha_{f(k)}) < 2^{-k}$$

for each $k \in \mathbb{N}$.

Suppose now S is a nonempty compact set in (X, d) . Then the density of α implies that for each $k \in \mathbb{N}$ there exists a nonempty finite subset A of $\{\alpha_i \mid i \in \mathbb{N}\}$ such that $d_H(S, A) < 2^{-k}$. This fact naturally leads to a definition of a computable (compact) set.

First, we will fix some effective enumeration of all nonempty finite subsets of \mathbb{N} . To do this, we will use the following notion.

Let $k, n \in \mathbb{N}$, $k, n \geq 1$, and $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$, where $\mathcal{P}(\mathbb{N}^n)$ denotes the power set of \mathbb{N}^n . We say that Φ is **computably finite valued** (c.f.v.) if

$$\{(x, y) \in \mathbb{N}^k \times \mathbb{N}^n \mid y \in \Phi(x)\}$$

is a computable subset of \mathbb{N}^{k+n} and there exists a computable function $\varphi : \mathbb{N}^k \rightarrow \mathbb{N}$ such that $\Phi(x) \subseteq \{0, \dots, \varphi(x)\}^n$ for each $x \in \mathbb{N}^k$.

From now on, let $\mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$,

$$(4) \quad j \mapsto [j],$$

be some fixed c.f.v. function whose image is the set of all nonempty finite subsets of \mathbb{N} (such a function certainly exists). Hence, $([j])_{j \in \mathbb{N}}$ is an effective enumeration of all nonempty finite subsets of \mathbb{N} .

Let (X, d, α) be a computable metric space. For $j \in \mathbb{N}$ we define

$$\Lambda_j = \{\alpha_i \mid i \in [j]\}.$$

Let S be a compact set in (X, d) . We say that S is a **computable set** in (X, d, α) if $S = \emptyset$ or there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$d_H(S, \Lambda_{f(k)}) < 2^{-k}$$

for each $k \in \mathbb{N}$. It is not hard to conclude that this definition does not depend on the choice of the function $([j])_{j \in \mathbb{N}}$ (see Proposition 2.3).

If (X, d, α) is a computable metric space, $i \in \mathbb{N}$ and r a positive rational number, then we say that $B(\alpha_i, r)$ is a **rational open ball** in (X, d, α) . Here, for $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball of radius r centered at x , i.e. $B(x, r) = \{y \in X \mid d(x, y) < r\}$. A finite union of rational open balls will be called a **rational open set** in (X, d, α) .

Let $q : \mathbb{N} \rightarrow \mathbb{Q}$ be some fixed computable function whose image is the set of all positive rational numbers and let $\tau_1, \tau_2 : \mathbb{N} \rightarrow \mathbb{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$.

Let (X, d, α) be a computable metric space. Let $(\lambda_i)_{i \in \mathbb{N}}$ be the sequence of points in X defined by $\lambda_i = \alpha_{\tau_1(i)}$ and let $(\rho_i)_{i \in \mathbb{N}}$ be the sequence of rational numbers defined by $\rho_i = q_{\tau_2(i)}$. For $i \in \mathbb{N}$ we define

$$(5) \quad I_i = B(\lambda_i, \rho_i).$$

Note that $\{I_i \mid i \in \mathbb{N}\}$ is the family of all open rational balls in (X, d, α) . For $j \in \mathbb{N}$ we define

$$J_j = \bigcup_{i \in [j]} I_i.$$

Clearly $\{J_i \mid i \in \mathbb{N}\}$ is the family of all rational open sets in (X, d, α) .

A closed set S in (X, d) is said to be **computably enumerable** (c.e.) in (X, d, α) if the set $\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$ is c.e. A compact set S in (X, d) is said to be **semicomputable** in (X, d, α) if the set $\{j \in \mathbb{N} \mid S \subseteq J_j\}$ is c.e. It is not hard to see that these definitions do not depend on the choice of the functions q , τ_1 , τ_2 and $([j])_{j \in \mathbb{N}}$.

We have the following characterization of a computable set (Proposition 2.6 in [10]):

$$(6) \quad S \text{ computable in } (X, d, \alpha) \Leftrightarrow S \text{ c.e. and semicomputable in } (X, d, \alpha).$$

Let (X, d, α) be a computable metric space and $x \in X$. If x is a computable point in (X, d, α) , then there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that (3)

holds. Since for all $a, b, c \in X$ we have $|d(a, c) - d(b, c)| \leq d(a, b)$, for all $i, k \in \mathbb{N}$ we have

$$|d(x, \alpha_i) - d(\alpha_{f(k)}, \alpha_i)| \leq d(x, \alpha_{f(k)}) < 2^{-k}$$

and it follows from Proposition 2.1(ii) that the function $\mathbb{N} \rightarrow \mathbb{R}, i \mapsto d(x, \alpha_i)$ is computable. Thus the function $\mathbb{N} \rightarrow \mathbb{R}, i \mapsto d(x, \lambda_i)$ is also computable. For $i \in \mathbb{N}$ we have

$$x \in I_i \Leftrightarrow d(x, \lambda_i) < \rho_i$$

and Proposition 2.1(iii) implies that the set $\{i \in \mathbb{N} \mid x \in I_i\}$ is c.e.

Conversely, if the set $\{i \in \mathbb{N} \mid x \in I_i\}$ is c.e., then the set $\Omega = \{(k, i) \in \mathbb{N}^2 \mid x \in I_i \text{ and } \rho_i < 2^{-k}\}$ is also c.e. and since for each $k \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $(k, i) \in \Omega$, there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $(k, f(k)) \in \Omega$ for each $k \in \mathbb{N}$. So $d(x, \lambda_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$ and it follows that x is a computable point. We have the following conclusion:

(7) x computable point in $(X, d, \alpha) \Leftrightarrow \{i \in \mathbb{N} \mid x \in I_i\}$ c.e. set.

Let (X, d, α) be a computable metric space and let $S \subseteq X$. We say that S is a **co-computably enumerable** (co-c.e.) set in (X, d, α) if there exists a c.e. set $A \subseteq \mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in A} I_i.$$

We say that S is a **computable closed** set if S is both c.e. and co-c.e.

Each computable set is a computable closed set [10]. Conversely, a computable closed set need not be computable even if it is compact. However, if (X, d, α) has the effective covering property (for the definition see [2]) and compact closed balls, then for compact sets the notions “computable” and “computable closed” coincide (Proposition 3.6 in [4]). This in particular holds in the previously described computable metric space $(\mathbb{R}^n, d, \alpha)$.

2.3. Formal properties. Let (X, d) be a metric space, $x, y \in X$ and $r, s > 0$. If $d(x, y) \geq r + s$, then $B(x, r) \cap B(y, s) = \emptyset$. Conversely, if $B(x, r) \cap B(y, s) = \emptyset$, then inequality $d(x, y) \geq r + s$ holds if (X, d) is Euclidean space, but it does not hold in general (for example, if d is the discrete metric). Nevertheless, we will use this inequality (actually, the strong inequality) to introduce certain relation of formal disjointness between rational open balls I_i and I_j (actually between the numbers i and j) in a computable metric space.

Similarly, if $d(x, y) + s \leq r$, then $B(y, s) \subseteq B(x, r)$, but the converse of this statement does not hold in general. Although this inequality does not characterize the fact that $B(y, s) \subseteq B(x, r)$, it will be useful for us in computable metric spaces to introduce certain notion of formal inclusion.

Let (X, d, α) be a computable metric space. Let $i, j \in \mathbb{N}$. (Recall the definition (5).) We say that I_i and I_j are **formally disjoint** and write $I_i \diamond I_j$ if

$$d(\lambda_i, \lambda_j) > \rho_i + \rho_j.$$

We say that I_i is **formally contained** in I_j and write $I_i \subseteq_F I_j$ if

$$d(\lambda_i, \lambda_j) + \rho_i < \rho_j.$$

The main properties of these two relations are stated in the next proposition.

Proposition 2.2. *Let (X, d, α) be a computable metric space. Then the sets*

$$(8) \quad \{(i, j) \in \mathbb{N}^2 \mid I_i \diamond I_j\} \text{ and } \{(i, j) \in \mathbb{N}^2 \mid I_i \subseteq_F I_j\}$$

are c.e. Furthermore, the following holds:

- (1) *if $i, j \in \mathbb{N}$ are such that $I_i \diamond I_j$, then $I_i \cap I_j = \emptyset$;*
- (2) *if $i, j \in \mathbb{N}$ are such that $I_i \subseteq_F I_j$, then $I_i \subseteq I_j$;*
- (3) *if $x, y \in X$ are such that $x \neq y$, then there exist $i, j \in \mathbb{N}$ such that $x \in I_i$, $y \in I_j$ and $I_i \diamond I_j$;*
- (4) *if $i, j \in \mathbb{N}$ and $x \in I_i \cap I_j$, then there exists $k \in \mathbb{N}$ such that $x \in I_k$, $I_k \subseteq_F I_i$ and $I_k \subseteq_F I_j$; moreover, if $A \subseteq \{\alpha_i \mid i \in \mathbb{N}\}$ is a dense set in (X, d) , k can be chosen so that $\lambda_k \in A$;*
- (5) *if $i, j \in \mathbb{N}$ are such that $I_i \diamond I_j$, then $I_j \diamond I_i$;*
- (6) *if $i, j, k \in \mathbb{N}$ are such that $I_i \subseteq_F I_j$ and $I_j \subseteq_F I_k$, then $I_i \subseteq_F I_k$;*
- (7) *if $i, j, k \in \mathbb{N}$ are such that $I_k \subseteq_F I_i$ and $I_i \diamond I_j$, then $I_k \diamond I_j$.*

Proof. It follows from the definition of $I_i \diamond I_j$ and $I_i \subseteq_F I_j$ and Proposition 2.1 that the sets in (8) are c.e. Furthermore, claims (1) and (2) obviously hold.

Let us prove (3). Suppose $x, y \in X$, $x \neq y$. Let $r = \frac{d(x, y)}{4}$. Choose $k \in \mathbb{N}$ so that $q_k < r$ and $u, v \in \mathbb{N}$ so that

$$x \in B(\alpha_u, q_k) \text{ and } y \in B(\alpha_v, q_k).$$

There exist $i, j \in \mathbb{N}$ such that $(u, k) = (\tau_1(i), \tau_2(i))$ and $(v, k) = (\tau_1(j), \tau_2(j))$ and therefore $(\alpha_u, q_k) = (\lambda_i, \rho_i)$ and $(\alpha_v, q_k) = (\lambda_j, \rho_j)$. So

$$x \in I_i \text{ and } y \in I_j.$$

We claim that $I_i \diamond I_j$. Suppose the opposite. Then $d(\lambda_i, \lambda_j) \leq \rho_i + \rho_j$, i.e. $d(\alpha_u, \alpha_v) \leq 2q_k$. We have

$$d(x, y) \leq d(x, \alpha_u) + d(\alpha_u, \alpha_v) + d(\alpha_v, y) < q_k + 2q_k + q_k = 4q_k < 4r = d(x, y),$$

i.e. $d(x, y) < d(x, y)$, a contradiction. Hence $I_i \diamond I_j$.

Let us prove (4). Suppose $i, j \in \mathbb{N}$ and $x \in I_i \cap I_j$. Then $d(x, \lambda_i) < \rho_i$ and $d(x, \lambda_j) < \rho_j$. Choose $v \in \mathbb{N}$ such that

$$d(x, \lambda_i) + 2q_v < \rho_i \text{ and } d(x, \lambda_j) + 2q_v < \rho_j.$$

Choose $u \in \mathbb{N}$ so that $d(x, \alpha_u) < q_v$ and $\alpha_u \in A$.

Let $k \in \mathbb{N}$ be such that $(\alpha_u, q_v) = (\lambda_k, \rho_k)$. Then $x \in I_k$. Furthermore,

$$d(\lambda_k, \lambda_i) + \rho_k = d(\alpha_u, \lambda_i) + q_v \leq d(\alpha_u, x) + d(x, \lambda_i) + q_v < d(x, \lambda_i) + 2q_v < \rho_i.$$

Hence $I_k \subseteq_F I_i$. In the same way we get $I_k \subseteq_F I_j$.

Claim (5) is obvious. It is straightforward to check that (6) holds.

We now prove (7). Suppose $I_k \subseteq_F I_i$ and $I_i \diamond I_j$. Since $I_k \subseteq_F I_i$, we have

$$(9) \quad \rho_k < \rho_i - d(\lambda_i, \lambda_k).$$

We also have $\rho_i + \rho_j < d(\lambda_i, \lambda_j) \leq d(\lambda_i, \lambda_k) + d(\lambda_k, \lambda_j)$, so $\rho_i + \rho_j < d(\lambda_i, \lambda_k) + d(\lambda_k, \lambda_j)$ and therefore

$$(10) \quad \rho_i - d(\lambda_i, \lambda_k) < -\rho_j + d(\lambda_k, \lambda_j).$$

It follows from (9) and (10) that $\rho_k < -\rho_j + d(\lambda_k, \lambda_j)$, hence $I_k \diamond I_j$. \square

2.4. Final remarks. The following properties of c.f.v. functions will be useful.

- Proposition 2.3.** (1) If $\Phi, \Psi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ are c.f.v. functions, then the function $\mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$, $x \mapsto \Phi(x) \cup \Psi(x)$ is c.f.v.
 (2) If $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ and $\Psi : \mathbb{N}^l \rightarrow \mathcal{P}(\mathbb{N}^l)$ are c.f.v. functions, then the function $\mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^{n+l})$, $x \mapsto \Phi(x) \times \Psi(x)$ is c.f.v.
 (3) If $\Phi, \Psi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ are c.f.v. functions, then the sets $\{x \in \mathbb{N}^k \mid \Phi(x) = \Psi(x)\}$ and $\{x \in \mathbb{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$ are computable.
 (4) Let $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ and $\Psi : \mathbb{N}^n \rightarrow \mathcal{P}(\mathbb{N}^m)$ be c.f.v. functions. Let $\Lambda : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^m)$ be defined by

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z),$$

$x \in \mathbb{N}^k$. Then Λ is a c.f.v. function.

- (5) Let $\Phi : \mathbb{N}^k \rightarrow \mathcal{P}(\mathbb{N}^n)$ be c.f.v. and let $T \subseteq \mathbb{N}^n$ be c.e. Then the set $S = \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq T\}$ is c.e. \square

Let $\sigma : \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\eta : \mathbb{N} \rightarrow \mathbb{N}$ be some fixed computable functions with the following property: $\{(\sigma(j, 0), \dots, \sigma(j, \eta(j))) \mid j \in \mathbb{N}\}$ is the set of all finite nonempty sequences in \mathbb{N} . We use the following notation: $(j)_i$ instead of $\sigma(j, i)$ and \bar{j} instead of $\eta(j)$. Hence

$$\{((j)_0, \dots, (j)_{\bar{j}}) \mid j \in \mathbb{N}\}$$

is the set of all finite nonempty sequences in \mathbb{N} .

It follows from Proposition 2.3(4) that the function $\mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$, $j \mapsto \{(j)_i \mid 0 \leq i \leq \bar{j}\}$ is c.f.v. Clearly, the image of this function is the set of all nonempty finite subsets of \mathbb{N} . This means that we can take this function as an effective enumeration introduced by (4) and it will suitable for us to do so. Therefore, we assume that

$$(11) \quad [j] = \{(j)_i \mid 0 \leq i \leq \bar{j}\}.$$

for each $j \in \mathbb{N}$.

3. COMPUTABLE TOPOLOGICAL SPACES

Proposition 2.2 is a motivation for the next definition.

Let (X, \mathcal{T}) be a topological space and let (I_i) be a sequences in \mathcal{T} such that $\{I_i \mid i \in \mathbb{N}\}$ is a basis for the topology \mathcal{T} . A triple $(X, \mathcal{T}, (I_i))$ is said to be a **computable topological space** (see the definition of a SCT_2 space in [25]) if there exist c.e. subsets \mathcal{C} and \mathcal{D} of \mathbb{N}^2 with the following properties:

- (1) if $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{D}$, then $I_i \cap I_j = \emptyset$;
- (2) if $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{C}$, then $I_i \subseteq I_j$;
- (3) if $x, y \in X$ are such that $x \neq y$, then there exist $i, j \in \mathbb{N}$ such that $x \in I_i$, $y \in I_j$ and $(i, j) \in \mathcal{D}$;
- (4) if $i, j \in \mathbb{N}$ and $x \in I_i \cap I_j$, then there exists $k \in \mathbb{N}$ such that $x \in I_k$, $(k, i) \in \mathcal{C}$ and $(k, j) \in \mathcal{C}$.

In this case we say that \mathcal{C} and \mathcal{D} are **characteristic relations** for $(X, \mathcal{T}, (I_i))$.

Note the following: if $(X, \mathcal{T}, (I_i))$ is a computable topological space, then (X, \mathcal{T}) is a second countable Hausdorff space.

Let (X, d, α) be a computable metric space. Let \mathcal{T}_d denote the topology induced by d , i.e. the set of all open sets in (X, d) . Let, for $i \in \mathbb{N}$, the set I_i be defined by (5) (for fixed functions q , τ_1 and τ_2). Then $\{I_i \mid i \in \mathbb{N}\}$ is a basis for the

topology \mathcal{T}_d and by Proposition 2.2 $(X, \mathcal{T}_d, (I_i))$ is a computable topological space; characteristic relations are $\{(i, j) \in \mathbb{N}^2 \mid I_i \subseteq_F I_j\}$ and $\{(i, j) \in \mathbb{N}^2 \mid I_i \diamond I_j\}$. We say that $(X, \mathcal{T}_d, (I_i))$ is the computable topological space associated to (X, d, α) .

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. Let $x \in X$. We say that x is a **computable point** in $(X, \mathcal{T}, (I_i))$ if the set $\{i \in \mathbb{N} \mid x \in I_i\}$ is c.e.

A closed set S in (X, \mathcal{T}) is said to be **computably enumerable** in $(X, \mathcal{T}, (I_i))$ if $\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$ is a c.e. set.

If $(X, \mathcal{T}, (I_i))$ is a computable topological space, then for $j \in \mathbb{N}$ we define J_j by

$$J_j = \bigcup_{i \in [j]} I_i.$$

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S be a compact set in (X, \mathcal{T}) . We say that S is a **semicomputable set** in $(X, \mathcal{T}, (I_i))$ if $\{j \in \mathbb{N} \mid S \subseteq J_j\}$ is a c.e. set. We say that S is a **computable set** in $(X, \mathcal{T}, (I_i))$ if S is computably enumerable and semicomputable in $(X, \mathcal{T}, (I_i))$. These definitions are easily seen to be independent on the choice of the function $([j])_{j \in \mathbb{N}}$.

Proposition 3.1. *Let (X, d, α) be a computable metric space and let $(X, \mathcal{T}_d, (I_i))$ be the associated computable topological space. Let $x \in X$ and $S \subseteq X$. The following equivalences hold:*

- (i) x computable point in $(X, d, \alpha) \Leftrightarrow x$ computable point in $(X, \mathcal{T}_d, (I_i))$;
- (ii) S c.e. set in $(X, d, \alpha) \Leftrightarrow S$ c.e. set in $(X, \mathcal{T}_d, (I_i))$;
- (iii) S semicomputable set in $(X, d, \alpha) \Leftrightarrow S$ semicomputable set in $(X, \mathcal{T}_d, (I_i))$;
- (iii) S computable set in $(X, d, \alpha) \Leftrightarrow S$ computable set in $(X, \mathcal{T}_d, (I_i))$.

Proof. This follows from (7) and (6). □

In this paper we prove that in any computable topological space $(X, \mathcal{T}, (I_i))$ the implication

$$S \text{ semicomputable} \Rightarrow S \text{ computable}$$

holds if S is, as a subspace of (X, \mathcal{T}) , a manifold whose boundary is computable. By Proposition 3.1 this is a generalization of the result from [10] for semicomputable manifolds in computable metric spaces.

Regarding the definition of a computable topological space, the natural question is this: if $(X, \mathcal{T}, (I_i))$ is a computable topological space, do there exist d and α such that (X, d, α) is a computable metric space whose associated computable topological space is $(X, \mathcal{T}, (I_i))$? In the following example we get that the answer is negative: (X, \mathcal{T}) need not be metrizable, moreover it need not be even regular (recall that (X, \mathcal{T}) is always second countable Hausdorff). The example is motivated by a classical example of a Hausdorff space which is not regular (see [5]).

Example 3.2. Let $c \in \mathbb{R} \setminus \mathbb{Q}$ be a computable number. Let $\beta : \mathbb{N} \rightarrow \mathbb{Q}$ be a computable surjection and let $\gamma : \mathbb{N} \rightarrow \mathbb{R}$ be defined by $\gamma(i) = c + \beta(i)$. Then γ is a computable function.

Let $\alpha : \mathbb{N} \rightarrow \mathbb{R}$ be defined by

$$\alpha_i = \begin{cases} \beta_{\tau_1(i)} & \text{if } \tau_2(i) \in 2\mathbb{N}, \\ \gamma_{\tau_1(i)} & \text{if } \tau_2(i) \notin 2\mathbb{N}. \end{cases}$$

Then α is a computable function and $\{\alpha_i \mid i \in \mathbb{N}\} = \mathbb{Q} \cup (c + \mathbb{Q})$.

Let $X = \mathbb{Q} \cup (c + \mathbb{Q})$ and let d be the Euclidean metric on X . Then (X, d, α) is a computable metric space. Let the sequences (λ_i) , (ρ_i) and (I_i) for this computable metric space be defined in the standard way. For $i \in \mathbb{N}$ we define

$$B_i = (I_i \cap \mathbb{Q}) \cup \{\lambda_i\}.$$

Let $\mathcal{D} = \{(i, j) \in \mathbb{N}^2 \mid I_i \diamond I_j\}$. Then \mathcal{D} is a c.e. set and $(i, j) \in \mathcal{D}$ clearly implies $B_i \cap B_j = \emptyset$.

Suppose $x, y \in X$ are such that $x \neq y$. Then there exists $i, j \in \mathbb{N}$ such that $x \in B_i$, $y \in B_j$ and $(i, j) \in \mathcal{D}$. Namely, choose a positive rational number r such that $2r < d(x, y)$ and choose $i, j \in \mathbb{N}$ such that $(x, r) = (\lambda_i, \rho_i)$ and $(y, r) = (\lambda_j, \rho_j)$. Then i and j are the desired numbers.

Let

$$\mathcal{C} = \{(i, j) \in \mathbb{N}^2 \mid I_i \subseteq_F I_j \text{ and } (\lambda_i = \lambda_j \text{ or } (\lambda_i \neq \lambda_j \text{ and } \lambda_i \in \mathbb{Q}))\}.$$

In general, if $f, g : \mathbb{N}^k \rightarrow \mathbb{Q}$ are computable functions, then the set $\{x \in \mathbb{N}^k \mid f(x) = g(x)\}$ is computable. Therefore the sets $\{(i, j) \in \mathbb{N}^2 \mid \beta_i = \beta_j\}$ and $\{(i, j) \in \mathbb{N}^2 \mid \gamma_i = \gamma_j\}$ are computable and it follows that the set $\{(i, j) \in \mathbb{N}^2 \mid \alpha_i = \alpha_j\}$ is computable. The set $\{i \in \mathbb{N} \mid \alpha_i \in \mathbb{Q}\}$ is also computable and since $\lambda_i = \alpha_{\tau_1(i)}$ for each $i \in \mathbb{N}$ we conclude that \mathcal{C} is a c.e. set.

If $(i, j) \in \mathcal{C}$, then obviously $B_i \subseteq B_j$.

Suppose now that $i, j \in \mathbb{N}$ and $x \in B_i \cap B_j$. We claim that there exists $k \in \mathbb{N}$ such that $x \in B_k$, $(k, i) \in \mathcal{C}$ and $(k, j) \in \mathcal{C}$. We have two cases.

Case 1 : $x \neq \lambda_i$ or $x \neq \lambda_j$. Then $x \in \mathbb{Q}$ and $x \in I_i \cap I_j$. By Proposition 2.2(4) there exists $k \in \mathbb{N}$ such that $x \in I_k$, $I_k \subseteq_F I_i$, $I_k \subseteq_F I_j$ and $\lambda_k \in \mathbb{Q}$. It follows $x \in B_k$, $(k, i) \in \mathcal{C}$ and $(k, j) \in \mathcal{C}$.

Case 2 : $x = \lambda_i = \lambda_j$. Choose a positive rational number r such that $r < \rho_i$ and $r < \rho_j$. Let $k \in \mathbb{N}$ be such that $(x, r) = (\lambda_k, \rho_k)$. Then $I_k \subseteq_F I_i$ and $I_k \subseteq_F I_j$ and we conclude that $x \in B_k$, $(k, i) \in \mathcal{C}$ and $(k, j) \in \mathcal{C}$.

In particular, we have the following conclusion: if $i, j \in \mathbb{N}$ and $x \in B_i \cap B_j$, then there exists $k \in \mathbb{N}$ such that $x \in B_k$, $B_k \subseteq B_i$ and $B_k \subseteq B_j$. This, together with the obvious fact that $X = \bigcup_{i \in \mathbb{N}} B_i$, implies that there exists a (unique) topology \mathcal{T} on X such that $\{B_i \mid i \in \mathbb{N}\}$ is a basis for \mathcal{T} .

Then the triple $(X, \mathcal{T}, (I_i))$ is a computable topological space: its characteristic relations are \mathcal{C} and \mathcal{D} .

We claim that the topological space (X, \mathcal{T}) is not regular. We have that \mathbb{Q} is the union of all B_i such that $\lambda_i \in \mathbb{Q}$. Therefore $\mathbb{Q} \in \mathcal{T}$ and therefore $c + \mathbb{Q}$ is a closed set in (X, \mathcal{T}) . Clearly $0 \notin c + \mathbb{Q}$.

Suppose (X, \mathcal{T}) is regular. Then there exist disjoint sets $U, V \in \mathcal{T}$ such that $0 \in U$ and $c + \mathbb{Q} \subseteq V$. It follows that there exists $i \in \mathbb{N}$ such that $0 \in B_i \subseteq U$. Hence $0 \in I_i \cap \mathbb{Q} \subseteq U$. So there exists an open interval K in \mathbb{R} such that

$$(12) \quad K \cap \mathbb{Q} \subseteq U.$$

Choose $x \in \mathbb{Q}$ such that $c + x \in K$. Since $c + x \in V$, there exists $j \in \mathbb{N}$ such that $c + x \in B_j \subseteq V$ and we conclude that there exists an open interval L in \mathbb{R} such that $c + x \in L$ and $L \cap \mathbb{Q} \subseteq V$. This and (12) imply $(K \cap L) \cap \mathbb{Q} = \emptyset$. But this is impossible since $c + x \in K \cap L$: if two open intervals have a common point, then they have a common rational point.

So (X, \mathcal{T}) is not regular.

Suppose $(X, \mathcal{T}, (I_i))$ is a computable topological space and \mathcal{C} and \mathcal{D} are its characteristic relations such that, beside the properties (1)–(4) from the definition of characteristic relations, the following additional properties hold:

- (5) if $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{D}$, then $(j, i) \in \mathcal{D}$;
- (6) $(i, i) \in \mathcal{C}$ for each $i \in \mathbb{N}$ and if $i, j, k \in \mathbb{N}$ are such that $(i, j) \in \mathcal{C}$ and $(j, k) \in \mathcal{C}$, then $(i, k) \in \mathcal{C}$;
- (7) if $i, j, k \in \mathbb{N}$ are such that $(k, i) \in \mathcal{C}$ and $(i, j) \in \mathcal{D}$, then $(k, j) \in \mathcal{D}$.

Then we say that \mathcal{C} and \mathcal{D} are **proper characteristic relations** for $(X, \mathcal{T}, (I_i))$.

Every computable topological space has proper characteristic relations. This is the contents of the following proposition.

Proposition 3.3. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. Then there exist proper characteristic relations for $(X, \mathcal{T}, (I_i))$.*

Proof. We first show that there exist characteristic relations for $(X, \mathcal{T}, (I_i))$ which satisfy properties (5) and (6) above.

Let \mathcal{C} and \mathcal{D} be characteristic relations for $(X, \mathcal{T}, (I_i))$. We define

$$\mathcal{D}' = \mathcal{D} \cup \{(i, j) \mid (j, i) \in \mathcal{D}\}$$

and we define \mathcal{C}' as the set of all $(i, j) \in \mathbb{N}^2$ for which there exist $n \in \mathbb{N}$ and $a_0, \dots, a_n \in \mathbb{N}$ such that $i = a_0$, $j = a_n$ and $(a_l, a_{l+1}) \in \mathcal{C}$ for each $l < n$. Clearly, \mathcal{D}' is c.e. On the other hand, the set

$$\Omega = \{a \in \mathbb{N} \mid ((a)_l, (a)_{l+1}) \in \mathcal{C} \text{ for each } l < \bar{a}\}$$

is c.e. (recall the notation from Subsection 2.4) by Proposition 2.3(5) since $\Omega = \{a \in \mathbb{N} \mid \Phi(a) \subseteq \mathcal{C}\}$, where $\Phi : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N}^2)$ is the c.f.v. function defined by $\Phi(a) = \{((a)_l, (a)_{l+1}) \mid l < \bar{a}\}$ (Proposition 2.3(4)). We have

$$\mathcal{C}' = \{(i, j) \in \mathbb{N}^2 \mid \text{there exists } a \in \mathbb{N} \text{ such that } i = (a)_0, j = (a)_{\bar{a}} \text{ and } a \in \Omega\}$$

and therefore \mathcal{C}' is c.e.

If $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{D}'$, then clearly $I_i \cap I_j = \emptyset$ and if $(i, j) \in \mathcal{C}'$, then $I_i \subseteq I_j$. Since $\mathcal{D} \subseteq \mathcal{D}'$ and $\mathcal{C} \subseteq \mathcal{C}'$, properties (3) and (4) from the definition of characteristic relations are also satisfied for \mathcal{C}' and \mathcal{D}' . Hence these are characteristic relations for $(X, \mathcal{T}, (I_i))$. It is immediate from their definitions that \mathcal{D}' is symmetric and \mathcal{C}' is reflexive and transitive, so properties (5) and (6) above are satisfied.

Suppose now that we have characteristic relations \mathcal{C} and \mathcal{D} for $(X, \mathcal{T}, (I_i))$ which satisfy (5) and (6). We define

$$\mathcal{D}' = \{(i, j) \in \mathbb{N}^2 \mid \exists k, l \in \mathbb{N} \text{ such that } (i, k) \in \mathcal{C}, (j, l) \in \mathcal{C} \text{ and } (k, l) \in \mathcal{D}\}.$$

It is easy to check that \mathcal{C} and \mathcal{D}' are proper characteristic relations for $(X, \mathcal{T}, (I_i))$. \square

4. EFFECTIVE SEPARATION OF COMPACT SETS

In this section let $(X, \mathcal{T}, (I_i))$ be some fixed computable topological space and let \mathcal{C} and \mathcal{D} be its proper characteristic relations.

Lemma 4.1. *Suppose $n \in \mathbb{N}$, $i_0, \dots, i_n \in \mathbb{N}$ and $x \in I_{i_0} \cap \dots \cap I_{i_n}$. Then there exists $k \in \mathbb{N}$ such that $x \in I_k$ and $(k, i_0), \dots, (k, i_n) \in \mathcal{C}$.*

Proof. Using reflexivity and transitivity of \mathcal{C} and property (4) from the definition of a computable topological space, this follows easily by induction. \square

Let $i, a \in \mathbb{N}$. We say that I_i is **\mathcal{C} -contained** in J_a and write $I_i \subseteq_{\mathcal{C}} J_a$ if there exists $j \in [a]$ such that $(i, j) \in \mathcal{C}$. Obviously $I_i \subseteq_{\mathcal{C}} J_a$ implies $I_i \subseteq J_a$.

Let $a, b \in \mathbb{N}$. We say that J_a is **\mathcal{C} -contained** in J_b and write $J_a \subseteq_{\mathcal{C}} J_b$ if $I_i \subseteq_{\mathcal{C}} J_b$ for each $i \in [a]$. If $J_a \subseteq_{\mathcal{C}} J_b$, then clearly $J_a \subseteq J_b$. Note also that $J_a \subseteq_{\mathcal{C}} J_a$ for each $a \in \mathbb{N}$.

Proposition 4.2. *Suppose K is a nonempty compact set in (X, \mathcal{T}) and $a, b \in \mathbb{N}$ are such that $K \subseteq J_a \cap J_b$. Then there exists $c \in \mathbb{N}$ such that $K \subseteq J_c$, $J_c \subseteq_{\mathcal{C}} J_a$ and $J_c \subseteq_{\mathcal{C}} J_b$.*

Proof. Let $x \in K$. Then there exists $i \in [a]$ and $j \in [b]$ such that $x \in I_i \cap I_j$. By definition of computable topological space, there exists $k \in \mathbb{N}$ such that $x \in I_k$ and $(k, i), (k, j) \in \mathcal{C}$.

So for each $x \in K$ there exists $k_x \in \mathbb{N}$ such that $x \in I_{k_x}$, $I_{k_x} \subseteq_{\mathcal{C}} J_a$ and $I_{k_x} \subseteq_{\mathcal{C}} J_b$. Since $\{I_{k_x} \mid x \in K\}$ is an open cover of K , there exists $n \in \mathbb{N}$ and $x_0, \dots, x_n \in K$ such that

$$K \subseteq I_{k_{x_0}} \cup \dots \cup I_{k_{x_n}}.$$

Choose $c \in \mathbb{N}$ such that $[c] = \{k_{x_0}, \dots, k_{x_n}\}$. Then $K \subseteq J_c$, $J_c \subseteq_{\mathcal{C}} J_a$ and $J_c \subseteq_{\mathcal{C}} J_b$. \square

If $a, b, c \in \mathbb{N}$ are such that $J_a \subseteq_{\mathcal{C}} J_b$ and $J_b \subseteq_{\mathcal{C}} J_c$, then the transitivity of \mathcal{C} easily gives $J_a \subseteq_{\mathcal{C}} J_c$. Using this, we get the following consequence of Proposition 4.2.

Corollary 4.3. *Let K be a nonempty compact set in (X, \mathcal{T}) , $n \in \mathbb{N}$ and $a_0, \dots, a_n \in \mathbb{N}$ such that $K \subseteq J_{a_0} \cap \dots \cap J_{a_n}$. Then there exists $c \in \mathbb{N}$ such that $K \subseteq J_c$ and $J_c \subseteq_{\mathcal{C}} J_{a_0}, \dots, J_c \subseteq_{\mathcal{C}} J_{a_n}$.*

Let $i, a \in \mathbb{N}$. We say that I_i and J_a are **\mathcal{D} -disjoint** and write $I_i \diamond_{\mathcal{D}} J_a$ if $(i, j) \in \mathcal{D}$ for each $j \in [a]$. Note that $I_i \diamond_{\mathcal{D}} J_a$ implies $I_i \cap J_a = \emptyset$.

Lemma 4.4. *Suppose K is a nonempty compact set in (X, \mathcal{T}) and $x \in X \setminus K$. Then there exist $i, a \in \mathbb{N}$ such that $x \in I_i$, $K \subseteq J_a$ and $I_i \diamond_{\mathcal{D}} J_a$.*

Proof. Let $y \in K$. Since $x \neq y$, by definition of computable topological space there exist $i_y, j_y \in \mathbb{N}$ such that $x \in I_{i_y}$, $y \in I_{j_y}$ and $(i_y, j_y) \in \mathcal{D}$. We have that $\{I_{j_y} \mid y \in K\}$ is an open cover of K and therefore there exist $n \in \mathbb{N}$ and $y_0, \dots, y_n \in K$ such that

$$(13) \quad K \subseteq I_{j_{y_0}} \cup \dots \cup I_{j_{y_n}}.$$

On the other hand, $x \in I_{i_{y_0}} \cap \dots \cap I_{i_{y_n}}$ and by Lemma 4.1 there exists $k \in \mathbb{N}$ such that $x \in I_k$ and $(k, i_{y_0}), \dots, (k, i_{y_n}) \in \mathcal{C}$. Since $(i_{y_0}, j_{y_0}), \dots, (i_{y_n}, j_{y_n}) \in \mathcal{C}$, by property (7) from the definition of proper characteristic relations we have

$$(14) \quad (k, j_{y_0}), \dots, (k, j_{y_n}) \in \mathcal{D}.$$

Choose $a \in \mathbb{N}$ so that $[a] = \{j_{y_0}, \dots, j_{y_n}\}$. Then $K \subseteq J_a$ by (13) and $I_k \diamond_{\mathcal{D}} J_a$ by (14). Since $x \in I_k$, this proves the lemma. \square

Let $a, b \in \mathbb{N}$. We say that J_a and J_b are **\mathcal{D} -disjoint** and write $J_a \diamond_{\mathcal{D}} J_b$ if $(i, j) \in \mathcal{D}$ for all $i \in [a]$ and $j \in [b]$. Clearly, $J_a \diamond_{\mathcal{D}} J_b$ if and only if $I_i \diamond_{\mathcal{D}} J_b$ for each $i \in [a]$. Note that $J_a \diamond_{\mathcal{D}} J_b$ implies $J_a \cap J_b = \emptyset$.

The following Lemma is a consequence of property (7) from the definition of proper characteristic relations.

Lemma 4.5. *Let $i, a, b, c, d \in \mathbb{N}$.*

- (i) *If $I_i \diamond_{\mathcal{D}} J_a$ and $J_b \subseteq_C J_a$, then $I_i \diamond_{\mathcal{D}} J_b$;*
- (ii) *If $J_c \diamond_{\mathcal{D}} J_a$ and $J_b \subseteq_C J_a$, then $J_c \diamond_{\mathcal{D}} J_b$.*
- (iii) *If $J_c \diamond_{\mathcal{D}} J_a$, $J_b \subseteq_C J_a$ and $J_d \subseteq_C J_c$ then $J_d \diamond_{\mathcal{D}} J_b$.*

Lemma 4.6. *Let K and L be nonempty disjoint compact sets in (X, \mathcal{T}) . Then there exists $a, b \in \mathbb{N}$ such that $K \subseteq J_a$, $L \subseteq J_b$ and $J_a \diamond_{\mathcal{D}} J_b$.*

Proof. Let $x \in K$. By Lemma 4.4 there exist $i_x, c_x \in \mathbb{N}$ such that $x \in I_{i_x}$, $L \subseteq J_{c_x}$ and $I_{i_x} \diamond_{\mathcal{D}} J_{c_x}$. Compactness of K implies that there exist $x_0, \dots, x_n \in K$ such that

$$K \subseteq I_{i_{x_0}} \cup \dots \cup I_{i_{x_n}}.$$

We have $L \subseteq J_{c_{x_0}} \cap \dots \cap J_{c_{x_n}}$ and by Corollary 4.3 there exists $b \in \mathbb{N}$ such that $L \subseteq J_b$ and $J_b \subseteq_C J_{c_{x_0}}, \dots, J_b \subseteq_C J_{c_{x_n}}$. We have $I_{i_{x_0}} \diamond_{\mathcal{D}} J_{c_{x_0}}, \dots, I_{i_{x_n}} \diamond_{\mathcal{D}} J_{c_{x_n}}$ and Lemma 4.5(i) implies that

$$I_{i_{x_0}} \diamond_{\mathcal{D}} J_b, \dots, I_{i_{x_n}} \diamond_{\mathcal{D}} J_b.$$

If we choose $a \in \mathbb{N}$ such that $[a] = \{i_{x_0}, \dots, i_{x_n}\}$, then we have $K \subseteq J_a$, $L \subseteq J_b$ and $J_a \diamond_{\mathcal{D}} J_b$. \square

Theorem 4.7. *Let \mathcal{F} be a finite family of nonempty compact sets in (X, \mathcal{T}) . Let A be a finite subset of \mathbb{N} . Then for each $K \in \mathcal{F}$ we can select $i_K \in \mathbb{N}$ so that the following hold:*

- (i) *$K \subseteq J_{i_K}$ for each $K \in \mathcal{F}$;*
- (ii) *if $K, L \in \mathcal{F}$ are such that $K \cap L = \emptyset$, then $J_{i_K} \diamond_{\mathcal{D}} J_{i_L}$;*
- (iii) *if $K \in \mathcal{F}$ and $a \in A$ are such that $K \subseteq J_a$, then $J_{i_K} \subseteq_C J_a$.*

Proof. Let us first notice that each compact set in (X, \mathcal{T}) is contained in some J_j .

Let $K, L \in \mathcal{F}$. By Lemma 4.6 there exist $u_{(K,L)}, v_{(K,L)} \in \mathbb{N}$ such that $K \subseteq J_{u_{(K,L)}}$, $L \subseteq J_{v_{(K,L)}}$ and such that

$$(15) \quad J_{u_{(K,L)}} \diamond_{\mathcal{D}} J_{v_{(K,L)}} \text{ if } K \cap L = \emptyset.$$

Let $K \in \mathcal{F}$. Observe the numbers $u_{(K,L)}$ and $v_{(L,K)}$, where $L \in \mathcal{F}$, and the numbers $a \in A$ such that $K \subseteq J_a$. There are only finitely many such numbers and so by Corollary 4.3 there exists $i_K \in \mathbb{N}$ such that $K \subseteq J_{i_K}$ and $J_{i_K} \subseteq_C J_{u_{(K,L)}}$ for each $L \in \mathcal{F}$, $J_{i_K} \subseteq_C J_{v_{(L,K)}}$ for each $L \in \mathcal{F}$ and $J_{i_K} \subseteq_C J_a$ for each $a \in A$ such that $K \subseteq J_a$.

Then the numbers i_K , $K \in \mathcal{F}$, are the required numbers. Properties (i) and (iii) clearly hold and if $K, L \in \mathcal{F}$ are such that $K \cap L = \emptyset$, then from $J_{i_K} \subseteq_C J_{u_{(K,L)}}$, $J_{i_L} \subseteq_C J_{v_{(K,L)}}$ and (15) we get $J_{i_K} \diamond_{\mathcal{D}} J_{i_L}$ (Lemma 4.5(iii)). \square

Proposition 4.8. *Let*

$$\Omega_1 = \{(i, a) \in \mathbb{N}^2 \mid I_i \subseteq_C J_a\}, \quad \Omega_2 = \{(a, b) \in \mathbb{N}^2 \mid J_a \subseteq_C J_b\},$$

$$\Gamma_1 = \{(i, a) \in \mathbb{N}^2 \mid I_a \diamond_{\mathcal{D}} J_i\}, \quad \Gamma_2 = \{(a, b) \in \mathbb{N}^2 \mid J_a \diamond_{\mathcal{D}} J_b\}.$$

Then Ω_1 , Ω_2 , Γ_1 and Γ_2 are c.e. sets.

Proof. Let $i, a \in \mathbb{N}$. We have

$$(i, a) \in \Omega_1 \Leftrightarrow \text{there exists } j \in \mathbb{N} \text{ such that } (i, j) \in \mathcal{C} \text{ and } j \in [a].$$

The set $\{(j, a) \in \mathbb{N}^2 \mid j \in [a]\}$ is computable and so Ω_1 is c.e.

The function $\Phi : \mathbb{N}^2 \rightarrow \mathcal{P}(\mathbb{N}^2)$ defined by $\Phi(a, b) = [a] \times \{b\}$ is c.f.v. by Proposition 2.3(2). For all $a, b \in \mathbb{N}$ we have

$$(a, b) \in \Omega_2 \Leftrightarrow I_i \subseteq_C J_b \text{ for each } i \in [a] \Leftrightarrow \Phi(a, b) \subseteq \Omega_1$$

and it follows from Proposition 2.3(5) that Ω_2 is c.e.

In a similar way we get that Γ_1 and Γ_2 are c.e. □

5. LOCAL COMPUTABLE ENUMERABILITY

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let A and S be subsets of X such that $A \subseteq S$. We say that A is **computably enumerable up to S** in $(X, \mathcal{T}, (I_i))$ if there exists a c.e. subset Ω of \mathbb{N} such that for each $i \in \mathbb{N}$ the following implications hold:

$$\begin{aligned} I_i \cap A \neq \emptyset &\implies i \in \Omega \\ i \in \Omega &\implies I_i \cap S \neq \emptyset. \end{aligned}$$

Note the following: if S is a closed set in (X, \mathcal{T}) , then S is c.e. in $(X, \mathcal{T}, (I_i))$ if and only if S is c.e. up to S in $(X, \mathcal{T}, (I_i))$.

Proposition 5.1. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let $A_0, \dots, A_n, S_0, \dots, S_n$ be subsets of X such that A_i is c.e. up to S_i for each $i \in \{0, \dots, n\}$. Then $A_0 \cup \dots \cup A_n$ is c.e. up to $S_0 \cup \dots \cup S_n$. In particular, if A_0, \dots, A_n are c.e. up to a set S , then $A_0 \cup \dots \cup A_n$ is c.e. up to S .*

Proof. Let $\Omega_0, \dots, \Omega_n$ be c.e. subsets of \mathbb{N} such that for each $j \in \{0, \dots, n\}$ and each $i \in \mathbb{N}$ the following implications hold:

$$(I_i \cap A_j \neq \emptyset \implies i \in \Omega_j) \text{ and } (i \in \Omega_j \implies I_i \cap S_j \neq \emptyset).$$

Then for each $i \in \mathbb{N}$ we have

$$I_i \cap (A_0 \cup \dots \cup A_n) \neq \emptyset \implies i \in \Omega_0 \cup \dots \cup \Omega_n$$

and

$$i \in \Omega_0 \cup \dots \cup \Omega_n \implies I_i \cap (S_0 \cup \dots \cup S_n) \neq \emptyset.$$

The set $\Omega_0 \cup \dots \cup \Omega_n$ is c.e. and the claim follows. □

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and $S \subseteq X$. Let $x \in S$. We say that S is **computably enumerable at x** in $(X, \mathcal{T}, (I_i))$ if there exists a neighborhood N of x in S such that N is c.e. up to S . We say that S is **locally computably enumerable** in $(X, \mathcal{T}, (I_i))$ if S is c.e. at x for each $x \in S$.

Each c.e. set in $(X, \mathcal{T}, (I_i))$ is clearly locally c.e.

Proposition 5.2. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S be a locally c.e. set in $(X, \mathcal{T}, (I_i))$ such that S is compact in (X, \mathcal{T}) . Then S is c.e. in $(X, \mathcal{T}, (I_i))$.*

Proof. For each $x \in S$ let N_x be a neighborhood of x in S such that N_x is c.e. up to S . The sets N_x , $x \in S$, are not necessarily open in S , but their interiors (in S) form an open cover of S and since S is compact, there exist $x_0, \dots, x_n \in S$ such that

$$(16) \quad S = N_{x_0} \cup \dots \cup N_{x_n}.$$

Each of the sets N_{x_0}, \dots, N_{x_n} is c.e. up to S and it follows from Proposition 5.1 and (16) that S is c.e. up to S . So S is c.e. (it is closed since it is compact and (X, \mathcal{T}) is Hausdorff). □

6. SEMICOMPUTABLE MANIFOLDS

In this section let $n \in \mathbb{N} \setminus \{0\}$ be fixed.

For $i \in \{1, \dots, n\}$ let

$$\begin{aligned} A_i &= \{(x_1, \dots, x_n) \in [-2, 2]^n \mid x_i = -2\}, \\ B_i &= \{(x_1, \dots, x_n) \in [-2, 2]^n \mid x_i = 2\}, \\ C_i &= \{(x_1, \dots, x_n) \in [-4, 4]^n \mid x_i \leq 1\}, \\ D_i &= \{(x_1, \dots, x_n) \in [-4, 4]^n \mid x_i \geq -1\}. \end{aligned}$$

We will use the following nontrivial topological fact (see Theorem 5.1 in [10], Corollary 3.2 in [9] and Theorem 1.8.1 in [6]).

Theorem 6.1. *Suppose U_1, \dots, U_n and V_1, \dots, V_n are open subsets of \mathbb{R}^n such that*

$$U_i \cap A_i = \emptyset, \quad V_i \cap B_i = \emptyset \quad \text{and} \quad U_i \cap V_i = \emptyset$$

for each $i \in \{1, \dots, n\}$. Then

$$[-2, 2]^n \not\subseteq U_1 \cup \dots \cup U_n \cup V_1 \cup \dots \cup V_n.$$

Lemma 6.2. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and S a semicomputable set in this space.*

- (i) *Let $m \in \mathbb{N}$. The set $S \setminus J_m$ is semicomputable in $(X, \mathcal{T}, (I_i))$.*
- (ii) *Let $k \in \mathbb{N} \setminus \{0\}$. The set $\{(j_1, \dots, j_k) \in \mathbb{N}^k \mid S \subseteq J_{j_1} \cup \dots \cup J_{j_k}\}$ is c.e.*

Proof. Claim (i) can be proved in the same way as Lemma 3.3 in [10]. For (ii), it is enough to prove that there exists a computable function $\varphi : \mathbb{N}^k \rightarrow \mathbb{N}$ such that $J_{j_1} \cup \dots \cup J_{j_k} = J_{\varphi(j_1, \dots, j_k)}$ for all $j_1, \dots, j_k \in \mathbb{N}$. For this, it is enough to prove that there exists a computable function $\varphi : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $J_a \cup J_b = J_{\varphi(a, b)}$ for all $a, b \in \mathbb{N}$. The function $\mathbb{N}^2 \rightarrow \mathcal{P}(\mathbb{N})$, $(a, b) \mapsto [a] \cup [b]$ is c.f.v. (Proposition 2.3(1)) and for all $a, b \in \mathbb{N}$ there exists $c \in \mathbb{N}$ such that $[a] \cup [b] = [c]$. The set $\{(a, b, c) \in \mathbb{N}^3 \mid [a] \cup [b] = [c]\}$ is computable (Proposition 2.3(3)) and therefore for all $a, b \in \mathbb{N}$ we can effectively find $c \in \mathbb{N}$ such that $[a] \cup [b] = [c]$. \square

In this paper we seek for conditions under which a semicomputable set is computable. Equivalently, we seek for conditions under which a semicomputable set is c.e. The next theorem is one of the main results of the paper. It gives a sufficient condition that a semicomputable set is c.e. at some point.

Theorem 6.3. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space, let S be a semicomputable set in this space and let $x \in S$. Suppose that there exists a neighborhood of x in S which is homeomorphic to some \mathbb{R}^n . Then S is c.e. at x .*

Proof. Let N be a neighborhood of x in S which is homeomorphic to \mathbb{R}^n . We may assume that N is open in S (as in the proof of Theorem 5.6 in [10]). Let $f : \mathbb{R}^n \rightarrow N$ be a homeomorphism. We may also assume that $f(0) = x$.

For $a, b \in \mathbb{R}$ we will denote by $\langle a, b \rangle$ the open interval $\{x \in \mathbb{R} \mid a < x < b\}$. The set $f(\langle -4, 4 \rangle^n)$ is open in N and therefore it is open in S . It follows that $S \setminus f(\langle -4, 4 \rangle^n)$ is compact (it is closed in the compact set S). This set is clearly disjoint with the compact set $f([-2, 2]^n)$ and Lemma 4.6 implies that there exists $m_0 \in \mathbb{N}$ such that

$$S \setminus f(\langle -4, 4 \rangle^n) \subseteq J_{m_0} \quad \text{and} \quad J_{m_0} \cap f([-2, 2]^n) = \emptyset.$$

Let $S' = S \setminus J_{m_0}$. By Lemma 6.2(i) S' is semicomputable in $(X, \mathcal{T}, (I_i))$ and we have

$$(17) \quad f([-2, 2]^n) \subseteq S' \subseteq f([-4, 4]^n).$$

Let $i \in \{1, \dots, n\}$. The sets A_i , B_i , C_i and D_i (defined at the begin of this section) are clearly compact in \mathbb{R}^n and we have $A_i \cap D_i = \emptyset$, $B_i \cap C_i = \emptyset$. Therefore $f(A_i)$, $f(B_i)$, $f(C_i)$ and $f(D_i)$ are compact in (X, \mathcal{T}) and $f(A_i) \cap f(D_i) = \emptyset$, $f(B_i) \cap f(C_i) = \emptyset$. By Lemma 4.6 there exist $d_i, c_i \in \mathbb{N}$ such that

$$(18) \quad f(C_i) \subseteq J_{c_i} \text{ and } J_{c_i} \cap f(B_i) = \emptyset,$$

$$(19) \quad f(D_i) \subseteq J_{d_i} \text{ and } J_{d_i} \cap f(A_i) = \emptyset.$$

Choose a computable function $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ such that $I_i = J_{\varphi(i)}$ for each $i \in \mathbb{N}$ (such a function certainly exists).

Let us assume that $l \in \mathbb{N}$ is such that

$$I_l \cap f([-1, 1]^n) \neq \emptyset.$$

Then there exists $v \in [-1, 1]^n$, $v = (v_1, \dots, v_n)$, such that $f(v) \in I_l$ and so $v \in f^{-1}(I_l)$. Since $f^{-1}(I_l)$ is open in \mathbb{R}^n , there exists $\epsilon > 0$ such that

$$(20) \quad [v_1 - \epsilon, v_1 + \epsilon] \times \dots \times [v_n - \epsilon, v_n + \epsilon] \subseteq f^{-1}(I_l).$$

We may assume $\epsilon < 1$. Let $E = [v_1 - \epsilon, v_1 + \epsilon] \times \dots \times [v_n - \epsilon, v_n + \epsilon]$. By (20) we have $f(E) \subseteq I_l$, i.e.

$$(21) \quad f(E) \subseteq J_{\varphi(l)}.$$

For $i \in \{1, \dots, n\}$ let

$$\tilde{A}_i = \{(x_1, \dots, x_n) \in [-4, 4]^n \mid x_i \leq v_i - \epsilon\},$$

$$\tilde{B}_i = \{(x_1, \dots, x_n) \in [-4, 4]^n \mid x_i \geq v_i + \epsilon\}.$$

Note that

$$(22) \quad \tilde{A}_i \subseteq C_i \text{ and } \tilde{B}_i \subseteq D_i$$

for each $i \in \{1, \dots, n\}$. Furthermore

$$\tilde{A}_1 \cup \tilde{B}_1 \cup \dots \cup \tilde{A}_n \cup \tilde{B}_n \cup E = [-4, 4]^n$$

and so

$$(23) \quad f(\tilde{A}_1) \cup f(\tilde{B}_1) \cup \dots \cup f(\tilde{A}_n) \cup f(\tilde{B}_n) \cup f(E) = f([-4, 4]^n).$$

For each $i \in \{1, \dots, n\}$ we have $\tilde{A}_i \cap \tilde{B}_i = \emptyset$, thus

$$(24) \quad f(\tilde{A}_i) \cap f(\tilde{B}_i) = \emptyset.$$

By (22) for each $i \in \{1, \dots, n\}$ we have $f(\tilde{A}_i) \subseteq f(C_i)$ and $f(\tilde{B}_i) \subseteq f(D_i)$ which, together with (18) and (19), gives

$$(25) \quad f(\tilde{A}_i) \subseteq J_{c_i} \text{ and } f(\tilde{B}_i) \subseteq J_{d_i}.$$

The sets $f(\tilde{A}_1), \dots, f(\tilde{A}_n), f(\tilde{B}_1), \dots, f(\tilde{B}_n), f(E)$ are nonempty and compact in (X, \mathcal{T}) . By Theorem 4.7, (24), (25) and (21) there exist numbers $a_1, \dots, a_n, b_1, \dots, b_n, e \in \mathbb{N}$ such that for each $i \in \{1, \dots, n\}$

$$f(\tilde{A}_i) \subseteq J_{a_i}, \quad f(\tilde{B}_i) \subseteq J_{b_i}, \quad f(E) \subseteq J_e, \\ J_{a_i} \subseteq_C J_{c_i}, \quad J_{b_i} \subseteq_C J_{d_i}, \quad J_e \subseteq_C J_{\varphi(l)} \text{ and } J_{a_i} \diamond_{\mathcal{D}} J_{b_i}.$$

It follows from (17) and (23) that

$$S' \subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e.$$

We have proved the following: if $l \in \mathbb{N}$ is such that $I_l \cap f([-1, 1]^n) \neq \emptyset$, then there exist $a_1, \dots, a_n, b_1, \dots, b_n, e \in \mathbb{N}$ such that

- (1) $J_{a_i} \subseteq_C J_{c_i}$ for each $i \in \{1, \dots, n\}$;
- (2) $J_{b_i} \subseteq_C J_{d_i}$ for each $i \in \{1, \dots, n\}$;
- (3) $J_e \subseteq_C J_{\varphi(l)}$
- (4) $J_{a_i} \diamond_{\mathcal{D}} J_{b_i}$ for each $i \in \{1, \dots, n\}$
- (5) $S' \subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e$.

Let Γ be the set of all $(l, a_1, \dots, a_n, b_1, \dots, b_n, e) \in \mathbb{N}^{2n+2}$ such that (1) - (5) hold. Furthermore, let Ω be the set of all $l \in \mathbb{N}$ for which there exist $a_1, \dots, a_n, b_1, \dots, b_n, e \in \mathbb{N}$ such that

$$(l, a_1, \dots, a_n, b_1, \dots, b_n, e) \in \Gamma.$$

Note that for each $l \in \mathbb{N}$ we have the following implication

$$I_l \cap f([-1, 1]^n) \neq \emptyset \implies l \in \Omega.$$

Using Proposition 4.8 and Lemma 6.2(2) we easily conclude that Γ is a c.e. set as the intersection of finitely many c.e. sets. It follows that Ω is also c.e.

We now prove the following: if $l \in \Omega$, then $I_l \cap S \neq \emptyset$.

Suppose $l \in \Omega$. Then there exist $a_1, \dots, a_n, b_1, \dots, b_n, e \in \mathbb{N}$ such that $(l, a_1, \dots, a_n, b_1, \dots, b_n, e) \in \Gamma$. So, for the numbers $l, a_1, \dots, a_n, b_1, \dots, b_n, e$ statements (1)–(5) hold.

Since $f([-2, 2]^n) \subseteq S'$, by (5) we have

$$f([-2, 2]^n) \subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e$$

and it follows

$$(26) \quad [-2, 2]^n \subseteq f^{-1}(J_{a_1}) \cup f^{-1}(J_{b_1}) \cup \dots \cup f^{-1}(J_{a_n}) \cup f^{-1}(J_{b_n}) \cup f^{-1}(J_e).$$

Let $i \in \{1, \dots, n\}$. It follows from (1) and (18) that $J_{a_i} \cap f(B_i) = \emptyset$ and therefore

$$(27) \quad f^{-1}(J_{a_i}) \cap B_i = \emptyset.$$

Similarly, from (2) and (19) we get

$$(28) \quad f^{-1}(J_{b_i}) \cap A_i = \emptyset.$$

By (4) we have

$$(29) \quad f^{-1}(J_{a_i}) \cap f^{-1}(J_{b_i}) = \emptyset.$$

The sets $f^{-1}(J_{a_1}), \dots, f^{-1}(J_{a_n}), f^{-1}(J_{b_1}), \dots, f^{-1}(J_{b_n})$ are open in \mathbb{R} . It follows from (27), (28), (29) and Theorem 6.1 that

$$[-2, 2]^n \not\subseteq f^{-1}(J_{a_1}) \cup f^{-1}(J_{b_1}) \cup \dots \cup f^{-1}(J_{a_n}) \cup f^{-1}(J_{b_n}).$$

This and (26) give

$$[-2, 2]^n \cap f^{-1}(J_e) \neq \emptyset.$$

So $f([-2, 2]^n) \cap J_e \neq \emptyset$ and (3) implies $f([-2, 2]^n) \cap J_{\varphi(l)} \neq \emptyset$. Hence $I_l \cap f([-2, 2]^n) \neq \emptyset$ and therefore $I_l \cap S \neq \emptyset$.

We have proved that for each $l \in \mathbb{N}$ the following implications hold:

- i) $I_l \cap f([-1, 1]^n) \neq \emptyset \implies l \in \Omega$
- ii) $l \in \Omega \implies I_l \cap S \neq \emptyset$.

Since $f([-1, 1]^n)$ is a neighborhood of x u S , this proves the theorem. \square

Let $n \in \mathbb{N} \setminus \{0\}$. A topological space X is said to be an n -**manifold** if each point $x \in X$ has a neighborhood in X which is homeomorphic to \mathbb{R}^n .

Theorem 6.4. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S be a semicomputable set in this space which is, as a subspace (X, \mathcal{T}) , a manifold. Then S is a computable set in $(X, \mathcal{T}, (I_i))$.*

Proof. By Theorem 6.3 S is locally c.e. By Proposition 5.2 S is c.e. So S is computable. \square

7. SEMICOMPUTABLE MANIFOLDS WITH COMPUTABLE BOUNDARIES

For $n \in \mathbb{N} \setminus \{0\}$ let

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

and

$$\text{Bd } \mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}.$$

A topological space X is said to be an n -**manifold with boundary** if for each $x \in X$ there exists a neighborhood N of x in X such that one of the following holds:

- (1) N is homeomorphic to \mathbb{R}^n ;
- (2) there exists a homeomorphism $f: \mathbb{H}^n \rightarrow N$ such that $x \in f(\text{Bd } \mathbb{H}^n)$.

If X is an n -manifold with boundary, we define ∂X to be the set of all $x \in X$ such that x has a neighborhood N with property (2). We say that ∂X is the **boundary** of the manifold X .

Each manifold is clearly a manifold with boundary. Conversely, if X is a manifold with boundary and $\partial X = \emptyset$, then X is a manifold.

It can be shown (see [18]) that if a point x in a topological space X has a neighborhood which satisfies (1), then it cannot have a neighborhood which satisfies (2). So a manifold with boundary X is a manifold if and only if $\partial X = \emptyset$.

In order to prove that a semicomputable manifold S with computable boundary is computable, we will need Theorem 6.3, but we will also need an analogue of this theorem which deals with points from ∂S (Theorem 7.2). First, we have a lemma.

Lemma 7.1. *Let $n \in \mathbb{N} \setminus \{0\}$. For $i \in \{1, \dots, n\}$ let*

$$B_i = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_i = 2\}.$$

For $i \in \{1, \dots, n-1\}$ let

$$A_i = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_i = -2\}$$

and let

$$A_n = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_n = 0\}.$$

Then there exist no open subsets $U_1, \dots, U_n, V_1, \dots, V_n$ of \mathbb{H}^n such that

$$(30) \quad U_i \cap B_i = \emptyset, \quad V_i \cap A_i = \emptyset \quad \text{and} \quad U_i \cap V_i = \emptyset$$

for each $i \in \{1, \dots, n\}$ and such that

$$(31) \quad [-2, 2]^{n-1} \times [0, 2] \subseteq U_1 \cup \dots \cup U_n \cup V_1 \cup \dots \cup V_n.$$

Proof. Suppose the opposite, i.e. suppose that there exist sets $U_1, \dots, U_n, V_1, \dots, V_n$ with the above properties.

Let $f: \mathbb{R} \rightarrow [0, \infty)$ be defined by

$$f(x) = \begin{cases} \frac{x+2}{2}, & x \geq -2 \\ 0, & x \leq -2. \end{cases}$$

Let $\gamma: \mathbb{R}^n \rightarrow \mathbb{H}^n$ be defined by

$$\gamma(z_1, \dots, z_{n-1}, z_n) = (z_1, \dots, z_{n-1}, f(z_n)).$$

Since f is continuous, γ is also continuous. We have

$$\gamma([-2, 2]^n) = [-2, 2]^{n-1} \times [0, 2].$$

From this and (31) it follows

$$(32) \quad [-2, 2]^n \subseteq \gamma^{-1}(U_1) \cup \dots \cup \gamma^{-1}(U_n) \cup \gamma^{-1}(V_1) \cup \dots \cup \gamma^{-1}(V_n).$$

By (30) for each $i \in \{1, \dots, n\}$ we have

$$(33) \quad \gamma^{-1}(U_i) \cap \gamma^{-1}(V_i) = \emptyset.$$

For $i \in \{1, \dots, n\}$ let

$$\tilde{A}_i = \{(x_1, \dots, x_n) \in [-2, 2]^n \mid x_i = -2\}$$

$$\tilde{B}_i = \{(x_1, \dots, x_n) \in [-2, 2]^n \mid x_i = 2\}.$$

Let $i \in \{1, \dots, n\}$. We have

$$\gamma(\tilde{B}_i) \subseteq B_i \quad \text{and} \quad \gamma(\tilde{A}_i) \subseteq A_i.$$

Since $U_i \cap B_i = \emptyset$, we have $U_i \cap \gamma(\tilde{B}_i) = \emptyset$ and consequently

$$(34) \quad \gamma^{-1}(U_i) \cap \tilde{B}_i = \emptyset.$$

Also $V_i \cap A_i = \emptyset$ implies

$$(35) \quad \gamma^{-1}(V_i) \cap \tilde{A}_i = \emptyset.$$

Since γ is continuous, the sets $\gamma^{-1}(U_1), \dots, \gamma^{-1}(U_n), \gamma^{-1}(V_1), \dots, \gamma^{-1}(V_n)$ are open in \mathbb{R}^n . This, together with (32), (33), (34) and (35) contradicts Theorem 6.1. \square

Theorem 7.2. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. Let S and T be semicomputable sets in this space such that $T \subseteq S$ and let $x \in S$. Let us suppose that there exists a neighborhood N of x in S and a homeomorphism $f: \mathbb{H}^n \rightarrow N$ (for some $n \in \mathbb{N}$) such that*

$$x \in f(\text{Bd } \mathbb{H}^n) \quad \text{and} \quad f(\text{Bd } \mathbb{H}^n) = N \cap T.$$

Then S is c.e. at x .

Proof. It is known that each open ball in \mathbb{H}^n (with respect to the Euclidean metric on \mathbb{H}^n) centered at a point in $\text{Bd } \mathbb{H}^n$ is homeomorphic to \mathbb{H}^n . Therefore, we may assume that N is an open neighborhood of x in S .

We may also assume that $x = f(0, \dots, 0)$.

As in the proof of Theorem 6.3 we conclude that the set $S \setminus f(\langle -4, 4 \rangle^{n-1} \times [0, 4])$ is compact in (X, \mathcal{T}) . The set $f([-2, 2]^{n-1} \times [0, 2])$ is also compact in (X, \mathcal{T}) . These two sets are disjoint and Lemma 4.6 implies that there exists $m_0 \in \mathbb{N}$ such that

$$(36) \quad S \setminus f(\langle -4, 4 \rangle^{n-1} \times [0, 4]) \subseteq J_{m_0} \quad \text{and} \quad J_{m_0} \cap f([-2, 2]^{n-1} \times [0, 2]) = \emptyset.$$

Let

$$S' = S \setminus J_{m_0},$$

$$T' = T \setminus J_{m_0}.$$

By Lemma 6.2(i) the sets S' and T' are semicomputable. We have

$$(37) \quad f([-2, 2]^{n-1} \times [0, 2]) \subseteq S' \subseteq f([-4, 4]^{n-1} \times [0, 4]).$$

For $i \in \{1, \dots, n-1\}$ let

$$C_i = \{(x_1, \dots, x_n) \in [-4, 4]^{n-1} \times [0, 4] \mid x_i \leq 1\},$$

$$D_i = \{(x_1, \dots, x_n) \in [-4, 4]^{n-1} \times [0, 4] \mid x_i \geq -1\},$$

$$A_i = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_i = -2\},$$

$$B_i = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_i = 2\}$$

and let

$$C_n = \{(x_1, \dots, x_n) \in [-4, 4]^{n-1} \times [0, 4] \mid x_n \leq 1\},$$

$$B_n = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_n = 2\}.$$

These sets are clearly compact in \mathbb{H}^n . Therefore, the sets $f(A_i)$, $f(B_i)$, $f(C_i)$, $f(D_i)$, for $i \in \{1, \dots, n-1\}$, and $f(C_n)$ and $f(B_n)$ are compact in (X, \mathcal{T}) . Moreover, for each $i \in \{1, \dots, n-1\}$ we have $f(A_i) \cap f(D_i) = \emptyset$ and $f(B_i) \cap f(C_i) = \emptyset$. Also $f(B_n) \cap f(C_n) = \emptyset$.

By Lemma 4.6 for each $i \in \{1, \dots, n-1\}$ there exist $d_i, c_i \in \mathbb{N}$ such that

$$(38) \quad f(C_i) \subseteq J_{c_i} \text{ and } J_{c_i} \cap f(B_i) = \emptyset,$$

$$(39) \quad f(D_i) \subseteq J_{d_i} \text{ and } J_{d_i} \cap f(A_i) = \emptyset$$

and there exists $c_n \in \mathbb{N}$ such that

$$(40) \quad f(C_n) \subseteq J_{c_n} \text{ and } J_{c_n} \cap f(B_n) = \emptyset.$$

Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ be a computable function such that $I_i = J_{\varphi(i)}$ for each $i \in \mathbb{N}$.

Suppose $l \in \mathbb{N}$ is such that

$$(41) \quad I_l \cap f([-1, 1]^{n-1} \times [0, 1]) \neq \emptyset.$$

Then

$$f^{-1}(I_l) \cap ([-1, 1]^{n-1} \times [0, 1]) \neq \emptyset$$

and since $f^{-1}(I_l)$ is open in \mathbb{H}^n we may choose $v \in [-1, 1]^{n-1} \times [0, 1]$, $v = (v_1, \dots, v_n)$, such that $v_n > 0$ and $v \in f^{-1}(I_l)$. The fact that $f^{-1}(I_l)$ is open implies that there exists $\epsilon > 0$ such that $\epsilon < v_n$ and

$$[v_1 - \epsilon, v_1 + \epsilon] \times \dots \times [v_n - \epsilon, v_n + \epsilon] \subseteq f^{-1}(I_l).$$

Let

$$E = [v_1 - \epsilon, v_1 + \epsilon] \times \dots \times [v_n - \epsilon, v_n + \epsilon].$$

So $E \subseteq f^{-1}(I_l)$ and $f(E) \subseteq I_l$.

For $i \in \{1, \dots, n\}$ let

$$\tilde{A}_i = \{(x_1, \dots, x_n) \in [-4, 4]^{n-1} \times [0, 4] \mid x_i \leq v_i - \epsilon\},$$

$$\tilde{B}_i = \{(x_1, \dots, x_n) \in [-4, 4]^{n-1} \times [0, 4] \mid x_i \geq v_i + \epsilon\}.$$

For each $i \in \{1, \dots, n-1\}$ we have

$$(42) \quad \tilde{A}_i \subseteq C_i, \tilde{B}_i \subseteq D_i \text{ and } \tilde{A}_n \subseteq C_n.$$

Furthermore, we have

$$\tilde{A}_1 \cup \tilde{B}_1 \cup \dots \cup \tilde{A}_n \cup \tilde{B}_n \cup E = [-4, 4]^{n-1} \times [0, 4]$$

and so

$$(43) \quad f(\tilde{A}_1) \cup f(\tilde{B}_1) \cup \dots \cup f(\tilde{A}_n) \cup f(\tilde{B}_n) \cup f(E) = f([-4, 4]^{n-1} \times [0, 4]).$$

For each $i \in \{1, \dots, n\}$ we have $f(\tilde{A}_i) \cap f(\tilde{B}_i) = \emptyset$ since $\tilde{A}_i \cap \tilde{B}_i = \emptyset$.

Let $i \in \{1, \dots, n-1\}$. It follows from (42) that $f(\tilde{A}_i) \subseteq f(C_i)$ and $f(\tilde{B}_i) \subseteq f(D_i)$ and so (38) and (39) imply

$$f(\tilde{A}_i) \subseteq J_{c_i} \text{ and } f(\tilde{B}_i) \subseteq J_{d_i}.$$

By (42) we have $f(\tilde{A}_n) \subseteq f(C_n)$ and from (40) we get $f(\tilde{A}_n) \subseteq J_{c_n}$.

We have

$$\begin{aligned} f(\tilde{B}_n) \cap T' &= (f(\tilde{B}_n) \cap N) \cap T' \\ &\subseteq (f(\tilde{B}_n) \cap N) \cap T \\ &= f(\tilde{B}_n) \cap (N \cap T) \\ &= f(\tilde{B}_n) \cap f(\text{Bd } \mathbb{H}^n) \\ &= f(\tilde{B}_n \cap \text{Bd } \mathbb{H}^n) \\ &= f(\emptyset) \\ &= \emptyset. \end{aligned}$$

Hence

$$(44) \quad f(\tilde{B}_n) \cap T' = \emptyset.$$

Let

$$A_n = \{(x_1, \dots, x_n) \in [-2, 2]^{n-1} \times [0, 2] \mid x_n = 0\}.$$

Since $A_n \subseteq \text{Bd } \mathbb{H}^n$, we have

$$f(A_n) \subseteq f(\text{Bd } \mathbb{H}^n) = N \cap T.$$

So $f(A_n) \subseteq T$. Furthermore $f(A_n) \cap J_{m_0} = \emptyset$ by (36). Therefore $f(A_n) \subseteq T \setminus J_{m_0}$, i.e.

$$(45) \quad f(A_n) \subseteq T'.$$

In particular, T' is a nonempty set. It follows from (44) and Lemma 4.6 that there exist $d_n, t \in \mathbb{N}$ such that $f(\tilde{B}_n) \subseteq J_{d_n}$, $T' \subseteq J_t$ and such that $J_{d_n} \diamond_{\mathcal{D}} J_t$.

The sets $\tilde{A}_1, \dots, \tilde{A}_n, \tilde{B}_1, \dots, \tilde{B}_n, E$ are nonempty and compact in \mathbb{H}^n . Consequently, the sets $f(\tilde{A}_1), \dots, f(\tilde{A}_n), f(\tilde{B}_1), \dots, f(\tilde{B}_n), f(E)$ are nonempty and compact in (X, \mathcal{T}) . Since $f(E) \subseteq I_l$, we have $f(E) \subseteq J_{\varphi(l)}$.

By Theorem 4.7 there exist $a_1, \dots, a_n, b_1, \dots, b_n, e \in \mathbb{N}$ such that for each $i \in \{1, \dots, n\}$ the following holds:

$$\begin{aligned} f(\tilde{A}_i) &\subseteq J_{a_i}, \quad f(\tilde{B}_i) \subseteq J_{b_i}, \quad f(E) \subseteq J_e, \\ J_{a_i} &\subseteq_{\mathcal{C}} J_{c_i}, \quad J_{b_i} \subseteq_{\mathcal{C}} J_{d_i}, \quad J_e \subseteq_{\mathcal{C}} J_{\varphi(l)} \text{ and } J_{a_i} \diamond_{\mathcal{D}} J_{b_i}. \end{aligned}$$

Since $J_{b_n} \subseteq_{\mathcal{C}} J_{d_n}$ and $J_{d_n} \diamond_{\mathcal{D}} J_t$, we have $J_{b_n} \diamond_{\mathcal{D}} J_t$.

It follows from (37) and (43) that

$$\begin{aligned} S' &\subseteq f(\tilde{A}_1) \cup f(\tilde{B}_1) \cup \dots \cup f(\tilde{A}_n) \cup f(\tilde{B}_n) \cup f(E) \\ &\subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e, \end{aligned}$$

hence

$$S' \subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e.$$

Let us summarize. If $l \in \mathbb{N}$ is such (41) holds, then there exist $a_1, \dots, a_n, b_1, \dots, b_n, e, t \in \mathbb{N}$ such that

- 1) $J_{a_i} \subseteq_C J_{c_i}$ for each $i \in \{1, \dots, n\}$
- 2) $J_{b_i} \subseteq_C J_{d_i}$ for each $i \in \{1, \dots, n-1\}$
- 3) $J_e \subseteq_C J_{\varphi(l)}$
- 4) $J_{a_i} \diamond_{\mathcal{D}} J_{b_i}$ for each $i \in \{1, \dots, n\}$
- 5) $T' \subseteq J_t$
- 6) $J_{b_n} \diamond_{\mathcal{D}} J_t$
- 7) $S' \subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e.$

Let Γ be the set of all $(l, a_1, \dots, a_n, b_1, \dots, b_n, e, t) \in \mathbb{N}^{2n+3}$ such that 1) - 7) hold. Furthermore, let Ω be the set of all $l \in \mathbb{N}$ for which there exist $a_1, \dots, a_n, b_1, \dots, b_n, e, t \in \mathbb{N}$ such that $(l, a_1, \dots, a_n, b_1, \dots, b_n, e, t) \in \Gamma$. We have proved the following:

if $l \in \mathbb{N}$ is such that $I_l \cap f([-1, 1]^{n-1} \times [0, 1]) \neq \emptyset$, then $l \in \Omega$.

Suppose now that $l \in \Omega$. Let us prove that

$$(46) \quad I_l \cap S \neq \emptyset.$$

Since $l \in \Omega$, there exist $a_1, \dots, a_n, b_1, \dots, b_n, e, t \in \mathbb{N}$ such that $(l, a_1, \dots, a_n, b_1, \dots, b_n, e, t) \in \Gamma$. So, for the numbers $l, a_1, \dots, a_n, b_1, \dots, b_n, e, t$ the statements 1) - 7) hold. By (37) and 7) we have

$$f([-2, 2]^{n-1} \times [0, 2]) \subseteq J_{a_1} \cup J_{b_1} \cup \dots \cup J_{a_n} \cup J_{b_n} \cup J_e$$

and it follows

$$(47) \quad [-2, 2]^{n-1} \times [0, 2] \subseteq f^{-1}(J_{a_1}) \cup f^{-1}(J_{b_1}) \cup \dots \cup f^{-1}(J_{a_n}) \cup f^{-1}(J_{b_n}) \cup f^{-1}(J_e).$$

Let $i \in \{1, \dots, n\}$. Then $J_{a_i} \subseteq J_{c_i}$ by 1) and it follows from (38) and (40) that $J_{a_i} \cap f(B_i) = \emptyset$. Therefore

$$(48) \quad f^{-1}(J_{a_i}) \cap B_i = \emptyset.$$

Let $i \in \{1, \dots, n-1\}$. It follows from 2) and (39) that $J_{b_i} \cap f(A_i) = \emptyset$ which gives

$$(49) \quad f^{-1}(J_{b_i}) \cap A_i = \emptyset.$$

By (45), 5) and 6) we have $J_{b_n} \cap f(A_n) = \emptyset$. Thus

$$f^{-1}(J_{b_n}) \cap A_n = \emptyset.$$

Statement 4) implies that

$$(50) \quad f^{-1}(J_{a_i}) \cap f^{-1}(J_{b_i}) = \emptyset$$

for each $i \in \{1, \dots, n\}$. The sets $f^{-1}(J_{a_1}), \dots, f^{-1}(J_{a_n}), f^{-1}(J_{b_1}), \dots, f^{-1}(J_{b_n})$ are clearly open in \mathbb{H}^n . From (48), (49), (50) and Lemma 7.1 we conclude that

$$[-2, 2]^{n-1} \times [0, 2] \not\subseteq f^{-1}(J_{a_1}) \cup \dots \cup f^{-1}(J_{a_n}) \cup f^{-1}(J_{b_1}) \cup \dots \cup f^{-1}(J_{b_n}).$$

From this and (47) we get

$$([-2, 2]^{n-1} \times [0, 2]) \cap f^{-1}(J_e) \neq \emptyset.$$

Hence $J_e \cap f([-2, 2]^{n-1} \times [0, 2]) \neq \emptyset$ which, together with 3) and $J_{\varphi(l)} = I_l$, gives

$$I_l \cap f([-2, 2]^{n-1} \times [0, 2]) \neq \emptyset.$$

This clearly implies (46).

We have proved that for each $l \in \mathbb{N}$ the following implications hold:

- i) $I_l \cap f([-1, 1]^{n-1} \times [0, 1]) \neq \emptyset \Rightarrow l \in \Omega$
- ii) $l \in \Omega \Rightarrow I_l \cap S \neq \emptyset$.

It is easy to conclude that Ω is a c.e. set. So, by i) and ii), the set $f([-1, 1]^{n-1} \times [0, 1])$ is c.e. up to S . Clearly $f([-1, 1]^{n-1} \times [0, 1])$ is a neighborhood of x in S . Hence S is c.e. at x . \square

The following theorem is a generalization of Theorem 6.4.

Theorem 7.3. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space, let $n \in \mathbb{N} \setminus \{0\}$ and let S be a semicomputable set in $(X, \mathcal{T}, (I_i))$ which has the following property: S is, as a subspace of (X, \mathcal{T}) , an n -manifold with boundary and ∂S is a semicomputable set in $(X, \mathcal{T}, (I_i))$. Then S is a computable set.*

Proof. Since S is compact, it suffices to prove that S is locally c.e.

Let $x \in S$. Then one of the following holds:

- 1) There exists a neighborhood of x in S which is homeomorphic to \mathbb{R}^n .
- 2) There exists a neighborhood N of x in S and a homeomorphism $f: \mathbb{H}^n \rightarrow N$ such that $x \in f(\text{Bd } \mathbb{H}^n)$.

If 1) holds, then S is c.e. at x by Theorem 6.3.

Suppose that 2) holds. We may assume that N is an open neighborhood of x . It is easy to conclude (see the proof of Theorem 6.1 in [10]) that

$$f(\text{Bd } \mathbb{H}^n) = N \cap \partial S.$$

Now Theorem 7.2 implies that S is c.e. at x .

So S is locally c.e. and the claim of the theorem follows. \square

8. COMPACTIFICATION AND SEMICOMPUTABILITY

If (X, d) is a metric space, for $x \in X$ and $r > 0$ by $\hat{B}(x, r)$ we denote the closed ball in (X, d) of radius r centered in x , i.e. $\hat{B}(x, r) = \{y \in X \mid d(y, x) \leq r\}$.

Let (X, d, α) be a computable metric space. If $p \in \mathbb{N}$ and r is a positive rational number, then we say that $\hat{B}(\alpha_p, r)$ is a **rational closed ball** in (X, d, α) . For $i \in \mathbb{N}$ we define

$$\hat{I}_i = \hat{B}(\lambda_i, \rho_i)$$

(recall the definition (5)). Then $\{\hat{I}_i \mid i \in \mathbb{N}\}$ is the family of all rational closed balls in (X, d, α) .

Semicomputable (compact) sets in a computable metric space can be characterized in the following way (see Proposition 3.1 in [4]).

Proposition 8.1. *Let (X, d, α) be a computable metric space and let S be a compact set in (X, d) . Then S is semicomputable in (X, d, α) if and only if $S \cap B$ is a compact set for each closed ball B in (X, d) and the set $\{(i, j) \in \mathbb{N}^2 \mid \hat{I}_i \cap S \subseteq J_j\}$ is c.e.*

Using this proposition, we extend the notion of a semicomputable set in a computable metric space to noncompact sets.

Let (X, d, α) be a computable metric space and let S be a subset of X (possibly noncompact). We say that S is **semicomputable** in (X, d, α) if the following holds (see the definition of a semi-c.c.b. set in [4]):

- (i) $S \cap B$ is a compact set for each closed ball B in (X, d) ;
- (ii) the set $\{(i, j) \in \mathbb{N}^2 \mid \hat{I}_i \cap S \subseteq J_j\}$ is c.e.

For a compact set S this definition, by Proposition 8.1, coincides with the earlier definition of a semicomputable set.

Condition (i) easily implies that each semicomputable set is closed.

In view of equivalence (6) we extend the notion of a computable set. If (X, d, α) is a computable metric space and $S \subseteq X$, then we say that S is **computable** if S is c.e. and semicomputable.

As before, we have that each computable set is a computable closed set (recall the definition of a computable closed set from Section 2). In computable metric spaces which have the effective covering property and compact closed balls, the notions “computable set” and “computable closed set” coincide [4].

Now it makes sense to ask does the implication

$$(51) \quad S \text{ semicomputable} \implies S \text{ computable}$$

hold for noncompact manifolds S (in a computable metric space)? In general, the answer is negative. It is not hard to construct a semicomputable 1-manifold in \mathbb{R}^2 which is not computable (see [4]). On the other hand, if S is a 1-manifold such that S has finitely many connected components, then (51) holds [4].

A general idea how to deal with the case when S is noncompact could be to apply certain construction which changes the ambient space and which turns S into a compact set (keeping the semicomputability of S). This construction, which is similar to a compactification of a space, leads to a new ambient space which is not a metric space, but a topological space and this is where the concept of a computable topological space will be applied.

Let us recall the notion of a one-point compactification. Suppose (X, \mathcal{T}) is a topological space and $Y = X \cup \{\infty\}$, where $\infty \notin X$. Let

$$\mathcal{S} = \mathcal{T} \cup \{\{\infty\} \cup U \mid U \in \mathcal{T} \text{ and } X \setminus U \text{ is compact in } (X, \mathcal{T})\}.$$

Then (Y, \mathcal{S}) is a compact topological space called a one-point compactification of (X, \mathcal{T}) .

Following the idea from this definition, we are going to use the following construction. Suppose (X, d, α) is a computable metric space and $Y = X \cup \{\infty\}$, where $\infty \notin X$. Let

$$(52) \quad \mathcal{S} = \mathcal{T}_d \cup \{\{\infty\} \cup U \mid U \text{ is open in } (X, d) \text{ and } X \setminus U \text{ is bounded in } (X, d)\}.$$

It is straightforward to check that (Y, \mathcal{S}) is a topological space and that (X, \mathcal{T}_d) is a subspace of (Y, \mathcal{S}) . For $i \in \mathbb{N}$ let

$$B_i = \begin{cases} I_{\frac{i}{2}}, & \text{if } i \in 2\mathbb{N} \\ \{\infty\} \cup \left(X \setminus \hat{I}_{\frac{i-1}{2}}\right), & \text{if } i \in 2\mathbb{N} + 1. \end{cases}$$

We say that the triple $(Y, \mathcal{S}, (B_i)_{i \in \mathbb{N}})$ is a **pseudocompactification** of the computable metric space (X, d, α) .

We claim that $(Y, \mathcal{S}, (B_i)_{i \in \mathbb{N}})$ is a computable topological space. First, we have the following lemma.

Lemma 8.2. *Let (X, d, α) be a computable metric space.*

- (i) *Let $x \in X$ and $i \in \mathbb{N}$ be such that $x \notin \hat{I}_i$. Then there exists $j \in \mathbb{N}$ such that $x \in I_j$ and $I_j \diamond I_i$.*
- (ii) *Let $i, j \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $I_i \subseteq_F I_k$ and $I_j \subseteq_F I_k$.*

Proof. (i) Since $x \notin \hat{I}_i$, we have $\rho_i < d(x, \lambda_i)$. Choose a positive rational number r such that $\rho_i + 2r < d(x, \lambda_i)$ and choose $k \in \mathbb{N}$ so that

$$(53) \quad d(\alpha_k, x) < r.$$

Then

$$(54) \quad d(\alpha_k, \lambda_i) > r + \rho_i.$$

Indeed, if $d(\alpha_k, \lambda_i) \leq r + \rho_i$, then

$$d(x, \lambda_i) \leq d(x, \alpha_k) + d(\alpha_k, \lambda_i) < r + r + \rho_i = 2r + \rho_i < d(x, \lambda_i),$$

a contradiction.

Choose $l \in \mathbb{N}$ so that $(\alpha_k, r) = (\lambda_j, \rho_j)$. Then, by (53), we have $x \in I_j$ and, by (54), $I_j \diamond I_i$.

- (ii) For any $n \in \mathbb{N}$ we can find a positive rational number r such that

$$d(\alpha_n, \lambda_i) + \rho_i < r \quad \text{and} \quad d(\alpha_n, \lambda_j) + \rho_j < r$$

and then a number $k \in \mathbb{N}$ such that $(\alpha_n, r) = (\lambda_k, \rho_k)$ is the desired number. \square

Theorem 8.3. *Let $(Y, \mathcal{S}, (B_i))$ be a pseudocompactification of a computable metric space (X, d, α) . Then $(Y, \mathcal{S}, (B_i))$ is a computable topological space.*

Proof. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$. We first show that \mathcal{B} is a basis for the topology \mathcal{S} . Clearly

$$\mathcal{B} = \{I_i \mid i \in \mathbb{N}\} \cup \left\{ \{\infty\} \cup (X \setminus \hat{I}_i) \mid i \in \mathbb{N} \right\}$$

and it is immediate that $\mathcal{B} \subseteq \mathcal{S}$.

Now we check that for each $V \in \mathcal{S}$ and each $x \in V$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq V$. Let $V \in \mathcal{S}$ and $x \in V$. We have two cases: $V \in \mathcal{T}_d$ and $V \notin \mathcal{T}_d$.

If $V \in \mathcal{T}_d$, then there exists $i \in \mathbb{N}$ such that $x \in I_i \subseteq V$ and clearly $I_i \in \mathcal{B}$.

Suppose $V \notin \mathcal{T}_d$. Then $V = \{\infty\} \cup U$, where U is open in (X, d) and $X \setminus U$ is bounded in (X, d) . We have $x \in \{\infty\} \cup U$. If $x \in U$, then there exists $i \in \mathbb{N}$ such that $x \in I_i \subseteq U$ and so $x \in I_i \subseteq V$.

Suppose $x = \infty$. Certainly, there exists $i \in \mathbb{N}$ such that $X \setminus U \subseteq \hat{I}_i$, which implies $X \setminus \hat{I}_i \subseteq U$ and we get

$$\infty \in \{\infty\} \cup (X \setminus \hat{I}_i) \subseteq \{\infty\} \cup U.$$

Hence, there exists $B \in \mathcal{B}$ such that $\infty \in B \subseteq V$. We conclude that \mathcal{B} is a basis for \mathcal{S} .

Let

$$\begin{aligned}\Gamma_1 &= \left\{ (i, j) \in \mathbb{N}^2 \mid i, j \in 2\mathbb{N} \text{ and } I_{\frac{i}{2}} \diamond I_{\frac{j}{2}} \right\}, \\ \Gamma_2 &= \left\{ (i, j) \in \mathbb{N}^2 \mid i \in 2\mathbb{N}, j \in 2\mathbb{N} + 1 \text{ and } I_{\frac{i}{2}} \subseteq_F I_{\frac{j-1}{2}} \right\}, \\ \Gamma_3 &= \left\{ (i, j) \in \mathbb{N}^2 \mid i \in 2\mathbb{N} + 1, j \in 2\mathbb{N} \text{ and } I_{\frac{j}{2}} \subseteq_F I_{\frac{i-1}{2}} \right\}, \\ \Gamma_4 &= \left\{ (i, j) \in \mathbb{N}^2 \mid i, j \in 2\mathbb{N} \text{ and } I_{\frac{i}{2}} \subseteq_F I_{\frac{j}{2}} \right\}, \\ \Gamma_5 &= \left\{ (i, j) \in \mathbb{N}^2 \mid i \in 2\mathbb{N}, j \in 2\mathbb{N} + 1 \text{ and } I_{\frac{i}{2}} \diamond I_{\frac{j-1}{2}} \right\}, \\ \Gamma_6 &= \left\{ (i, j) \in \mathbb{N}^2 \mid i, j \in 2\mathbb{N} + 1 \text{ and } I_{\frac{i-1}{2}} \subseteq_F I_{\frac{j-1}{2}} \right\}.\end{aligned}$$

Let

$$\mathcal{D} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

and

$$\mathcal{C} = \Gamma_4 \cup \Gamma_5 \cup \Gamma_6.$$

We claim that \mathcal{C} and \mathcal{D} are characteristic relations for $(Y, \mathcal{S}, (B_i)_{i \in \mathbb{N}})$.

Using Proposition 2.2 we conclude that the sets $\Gamma_1, \dots, \Gamma_6$ are c.e. So \mathcal{C} and \mathcal{D} are c.e. We now verify properties (1)-(4) from the definition of a computable topological space.

(1) Suppose $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{D}$.

Case 1 $(i, j) \in \Gamma_1$. Then $I_{\frac{i}{2}} \cap I_{\frac{j}{2}} = \emptyset$ and since $B_i = I_{\frac{i}{2}}$, $B_j = I_{\frac{j}{2}}$, we have $B_i \cap B_j = \emptyset$.

Case 2 $(i, j) \in \Gamma_2$. Then $I_{\frac{i}{2}} \subseteq I_{\frac{j-1}{2}}$, which implies $I_{\frac{i}{2}} \subseteq \hat{I}_{\frac{j-1}{2}}$ and therefore $I_{\frac{i}{2}} \cap (\{\infty\} \cup (X \setminus \hat{I}_{\frac{j-1}{2}})) = \emptyset$. So $B_i \cap B_j = \emptyset$.

Case 3 $(i, j) \in \Gamma_3$. In the same way we get $B_i \cap B_j = \emptyset$.

(2) Suppose $i, j \in \mathbb{N}$ are such that $(i, j) \in \mathcal{C}$.

Case 1 $(i, j) \in \Gamma_4$. Then $I_{\frac{i}{2}} \subseteq I_{\frac{j}{2}}$ and $B_i \subseteq B_j$.

Case 2 $(i, j) \in \Gamma_5$. Then $I_{\frac{i}{2}} \cap \hat{I}_{\frac{j-1}{2}} = \emptyset$ and so $I_{\frac{i}{2}} \subseteq (X \setminus \hat{I}_{\frac{j-1}{2}}) \cup \{\infty\}$. Hence, $B_i \subseteq B_j$.

Case 3 $(i, j) \in \Gamma_6$. Then $\hat{I}_{\frac{j-1}{2}} \subseteq I_{\frac{i-1}{2}}$, which implies $\hat{I}_{\frac{j-1}{2}} \subseteq \hat{I}_{\frac{i-1}{2}}$ and this gives $X \setminus \hat{I}_{\frac{j-1}{2}} \subseteq X \setminus \hat{I}_{\frac{i-1}{2}}$. So $B_i \subseteq B_j$.

(3) Suppose $x, y \in X \cup \{\infty\}$, $x \neq y$.

Case 1 $x, y \in X$. Then there exist $i, j \in \mathbb{N}$ such that $x \in I_i$, $y \in I_j$ and such that $I_i \diamond I_j$. It follows $x \in B_{2i}$, $y \in B_{2j}$ and $(2i, 2j) \in \mathcal{D}$.

Case 2 One of the points x and y is equal to ∞ . We may assume $y = \infty$. Then clearly $x \in X$. Choose $j \in \mathbb{N}$ such that $x \in I_j$. Then there exists $i \in \mathbb{N}$ such that $x \in I_i$ and $I_i \subseteq_F I_j$. It follows $x \in B_{2i}$, $\infty \in B_{2j+1}$ and $(2i, 2j+1) \in \mathcal{D}$.

(4) Suppose $i, j \in \mathbb{N}$ and $x \in B_i \cap B_j$.

Case 1 $i, j \in 2\mathbb{N}$. Then $x \in I_{\frac{i}{2}} \cap I_{\frac{j}{2}}$ and therefore there exists $k \in \mathbb{N}$ such that $x \in I_k$ and $I_k \subseteq_F I_{\frac{i}{2}}$ and $I_k \subseteq_F I_{\frac{j}{2}}$. So $x \in B_{2k}$, $(2k, i) \in \mathcal{C}$ and $(2k, j) \in \mathcal{C}$.

Case 2 $i \in 2\mathbb{N}$, $j \in 2\mathbb{N} + 1$. Then $x \in I_{\frac{i}{2}} \cap (\{\infty\} \cup (X \setminus \hat{I}_{\frac{j-1}{2}}))$. It follows $x \in I_{\frac{i}{2}}$ and $x \notin \hat{I}_{\frac{j-1}{2}}$. By Lemma 8.2 there exists $l \in \mathbb{N}$ such that $x \in I_l$ and $I_l \diamond I_{\frac{j-1}{2}}$. We have $x \in I_{\frac{i}{2}} \cap I_l$ and therefore there exists $k \in \mathbb{N}$ such that $x \in I_k$, $I_k \subseteq_F I_{\frac{i}{2}}$ and $I_k \subseteq_F I_l$. It follows $I_k \diamond I_{\frac{j-1}{2}}$. Hence we have $x \in B_{2k}$, $(2k, i) \in \mathcal{C}$ and $(2k, j) \in \mathcal{C}$.

Case 3 $i \in 2\mathbb{N} + 1$, $j \in 2\mathbb{N}$. This is essentially Case 2.

Case 4 $i, j \in 2\mathbb{N} + 1$. Then

$$x \in \left(\{\infty\} \cup \left(X \setminus \hat{I}_{\frac{i-1}{2}} \right) \right) \cap \left(\{\infty\} \cup \left(X \setminus \hat{I}_{\frac{j-1}{2}} \right) \right).$$

Subcase 1 $x \in X$. We have

$$x \notin \hat{I}_{\frac{i-1}{2}} \quad \text{and} \quad x \notin \hat{I}_{\frac{j-1}{2}}.$$

By Lemma 8.2 there exist $i', j' \in \mathbb{N}$ such that $x \in I_{i'}$, $I_{i'} \diamond I_{\frac{i-1}{2}}$, $x \in I_{j'}$ and $I_{j'} \diamond I_{\frac{j-1}{2}}$.

We have $x \in I_{i'} \cap I_{j'}$ and so there exists $k \in \mathbb{N}$ such that $x \in I_k$, $I_k \subseteq_F I_{i'}$ and $I_k \subseteq_F I_{j'}$. It follows $I_k \diamond I_{\frac{i-1}{2}}$ and $I_k \diamond I_{\frac{j-1}{2}}$. We have $x \in B_{2k}$, $(2k, i) \in \mathcal{C}$ and $(2k, j) \in \mathcal{C}$.

Subcase 2 $x = \infty$. By Lemma 8.2 there exists $k \in \mathbb{N}$ such that $I_{\frac{i-1}{2}} \subseteq_F I_k$ and $I_{\frac{j-1}{2}} \subseteq_F I_k$. We have $\infty \in B_{2k+1}$, $(2k+1, i) \in \mathcal{C}$ and $(2k+1, j) \in \mathcal{C}$.

We have proved that \mathcal{C} and \mathcal{D} are characteristic relations for $(Y, \mathcal{S}, (B_i)_{i \in \mathbb{N}})$. Hence $(Y, \mathcal{S}, (B_i)_{i \in \mathbb{N}})$ is a computable topological space. \square

If a metric space (X, d) has compact closed balls, then (Y, \mathcal{S}) , where \mathcal{S} is given by (52), is a one-point compactification of (X, \mathcal{T}_d) . Moreover, we have the following proposition.

Proposition 8.4. *Let (X, d) be a metric space, let $Y = X \cup \{\infty\}$, where $\infty \notin X$, and let \mathcal{S} be given by (52). Suppose $K \subseteq X$ is such that $K \cap D$ is a compact set in (X, d) for each closed ball D in (X, d) . Then $K \cup \{\infty\}$, as a subspace of (Y, \mathcal{S}) , is a one-point compactification of K (where K is taken as a subspace of (X, \mathcal{T}_d)). In particular, $K \cup \{\infty\}$ is a compact set in (Y, \mathcal{S}) .*

Proof. Let $V \subseteq K \cup \{\infty\}$. By the definition of the subspace topology, V is open in $K \cup \{\infty\}$ if and only if there exists an open set U in (X, d) such that

$$V = K \cap U \text{ or } (V = (K \cap U) \cup \{\infty\} \text{ and } X \setminus U \text{ bounded in } (X, d)).$$

Suppose U is open and $X \setminus U$ is bounded in (X, d) . Let $W = K \cap U$. Then W is open in K and $K \setminus W = K \setminus U$ is closed and bounded in K , which, together with the assumption of the proposition, gives that $K \setminus W$ is compact in K .

Conversely, if W is an open set in K such that $K \setminus W$ is compact in K , then $W = K \cap U$, where U is open in (X, d) . Since K is closed in (X, d) (which follows

from the assumption of the proposition), the set $U' = U \cup (X \setminus K)$ is open in (X, d) . We have $W = K \cap U'$ and

$$X \setminus U' = (X \setminus U) \cap K = K \setminus U = K \setminus W,$$

hence $X \setminus U'$ is bounded in (X, d) .

Altogether, we have the following conclusion: V is open in $K \cup \{\infty\}$ if and only if V is open in K or $V = W \cup \{\infty\}$, where W is open in K and $K \setminus W$ compact in K . \square

Let (X, d, α) be a computable metric space. For $l \in \mathbb{N}$ we define

$$L_l = \bigcap_{i \in [l]} \hat{I}_i.$$

Let $i, l \in \mathbb{N}$. We write

$$I_i \diamond L_l$$

if there exists $j \in [l]$ such that $I_i \diamond I_j$. Note: if $I_i \diamond L_l$, then $I_i \cap L_l = \emptyset$.

Let $u, l \in \mathbb{N}$. We write

$$J_u \diamond L_l$$

if $I_i \diamond L_l$ for each $i \in [u]$. Note: if $J_u \diamond L_l$, then $J_u \cap L_l = \emptyset$.

The following proposition can be proved in the same fashion as Proposition 4.8.

Proposition 8.5. *Let (X, d, α) be a computable metric space. Then the sets*

$$\{(i, l) \in \mathbb{N}^2 \mid I_i \diamond L_l\} \text{ and } \{(u, l) \in \mathbb{N}^2 \mid J_u \diamond L_l\}$$

are c.e.

Lemma 8.6. *Let (X, d, α) be a computable metric space.*

- (i) *Let $l \in \mathbb{N}$ and $x \in X$ be such that $x \notin L_l$. Then there exists $i \in \mathbb{N}$ such that $x \in I_i$ and $I_i \diamond L_l$.*
- (ii) *Let $l \in \mathbb{N}$ and let K be a nonempty compact set in (X, d) such that $K \cap L_l = \emptyset$. Then there exists $u \in \mathbb{N}$ such that $J_u \diamond L_l$ and $K \subseteq J_u$.*

Proof. (i) Since $x \notin L_l$, there exists $j \in [l]$ such that $x \notin \hat{I}_j$. By Lemma 8.2 there exists $i \in \mathbb{N}$ such that $x \in I_i$ and $I_i \diamond I_j$ and it follows $I_i \diamond L_l$.

(ii) Using (i) and the compactness of K we conclude that there exist $i_0, \dots, i_n \in \mathbb{N}$ such that

$$K \subseteq I_{i_0} \cup \dots \cup I_{i_n}$$

and $I_{i_0} \diamond L_l, \dots, I_{i_n} \diamond L_l$. Now we take $u \in \mathbb{N}$ such that $[u] = \{i_0, \dots, i_n\}$. \square

Proposition 8.7. *Let (X, d, α) be a computable metric space and let S be a semi-computable set in (X, d, α) . Then the set*

$$\{(l, j) \in \mathbb{N}^2 \mid S \cap L_l \subseteq J_j\}$$

is c.e.

Proof. Let $l, j \in \mathbb{N}$. We claim that

$$(55) \quad S \cap L_l \subseteq J_j$$

if and only if

$$(56) \quad S \cap \hat{I}_{(l)_0} \subseteq J_j \text{ or } (\exists u \in \mathbb{N} \text{ such that } J_u \diamond L_l \text{ and } S \cap \hat{I}_{(l)_0} \subseteq J_j \cup J_u)$$

(recall the notation from Subsection 2.4).

Let us suppose that (55) holds. The set $S \cap \hat{I}_{(l)_0}$ is closed since it is compact. Therefore $(S \cap \hat{I}_{(l)_0}) \setminus J_j$ is closed and, as a subset of a compact set $S \cap \hat{I}_{(l)_0}$, it is also compact.

If $x \in (S \cap \hat{I}_{(l)_0}) \setminus J_j$, then $x \in S$ and $x \notin J_j$, which, together with (55), implies $x \notin L_l$. This means that $((S \cap \hat{I}_{(l)_0}) \setminus J_j) \cap L_l = \emptyset$.

If $(S \cap \hat{I}_{(l)_0}) \setminus J_j = \emptyset$, then obviously $S \cap \hat{I}_{(l)_0} \subseteq J_j$.

Suppose $(S \cap \hat{I}_{(l)_0}) \setminus J_j \neq \emptyset$. By Lemma 8.6 there exists $u \in \mathbb{N}$ such that $J_u \diamond L_l$ and $(S \cap \hat{I}_{(l)_0}) \setminus J_j \subseteq J_u$. It follows $S \cap \hat{I}_{(l)_0} \subseteq J_j \cup J_u$.

Hence, (55) implies (56).

Suppose now that (56) holds. If $S \cap \hat{I}_{(l)_0} \subseteq J_j$, then from $L_l \subseteq \hat{I}_{(l)_0}$ it follows $S \cap L_l \subseteq J_j$. If there exists $u \in \mathbb{N}$ such that $J_u \diamond L_l$ and $S \cap \hat{I}_{(l)_0} \subseteq J_j \cup J_u$, then we have $J_u \cap L_l = \emptyset$ and it follows $S \cap L_l \subseteq J_j$.

So the statements (55) and (56) are equivalent. Using Lemma 6.2, Proposition 8.5 and the fact that S is semicomputable it is easy to conclude that the set of all $(l, j) \in \mathbb{N}^2$ for which (56) holds is c.e. This proves the claim of the proposition. \square

The main idea about pseudocompactifications is to reduce the problem of computability of noncompact semicomputable sets in (X, d, α) to computability of (compact) semicomputable sets in $(Y, \mathcal{S}, (B_i))$. Note the following: if the metric space (X, d) is bounded, each semicomputable set in (X, d, α) is compact. Therefore, the case when (X, d) is bounded is not interesting in view of pseudocompactifications.

Proposition 8.8. *Let (X, d, α) be a computable metric space and let $(Y, \mathcal{S}, (B_i))$ be its pseudocompactification. Let K be a semicomputable set in (X, d, α) . Suppose the metric space (X, d) is unbounded.*

- (i) *If K is compact in (X, d) , then K is semicomputable in $(Y, \mathcal{S}, (B_i))$.*
- (ii) *If K is not compact in (X, d) , then $K \cup \{\infty\}$ is semicomputable in $(Y, \mathcal{S}, (B_i))$.*

Proof. For $j \in \mathbb{N}$ let

$$C_j = B_{(j)_0} \cup \dots \cup B_{(j)_{\bar{j}}}.$$

Let $\Phi, \Psi: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$ be defined by

$$\Phi(j) = [j] \cap 2\mathbb{N} \text{ and } \Psi(j) = [j] \cap (2\mathbb{N} + 1).$$

These functions are clearly c.f.v. Let $j \in \mathbb{N}$. We have $[j] = \Phi(j) \cup \Psi(j)$ and

$$(57) \quad C_j = \bigcup_{i \in [j]} B_i = \bigcup_{i \in \Phi(j)} B_i \cup \bigcup_{i \in \Psi(j)} B_i = \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}} \cup \bigcup_{i \in \Psi(j)} \left(\{\infty\} \cup \left(X \setminus \hat{I}_{\frac{i-1}{2}} \right) \right).$$

(ii) Suppose K is not compact in (X, d) . We want to prove that $K \cup \{\infty\}$ is semicomputable in $(Y, \mathcal{S}, (B_i))$. By Proposition 8.4 $K \cup \{\infty\}$ is compact in (Y, \mathcal{S}) , so it remains to prove that the set

$$(58) \quad \{j \in \mathbb{N} \mid K \cup \{\infty\} \subseteq C_j\}$$

is c.e.

Let $j \in \mathbb{N}$. Using (57) we get

$$\begin{aligned} K \cup \{\infty\} \subseteq C_j &\Leftrightarrow \Psi(j) \neq \emptyset \text{ and } K \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}} \cup \bigcup_{i \in \Psi(j)} \left(X \setminus \hat{I}_{\frac{i-1}{2}} \right) \\ &\Leftrightarrow \Psi(j) \neq \emptyset \text{ and } K \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}} \cup \left(X \setminus \left(\bigcap_{i \in \Psi(j)} \hat{I}_{\frac{i-1}{2}} \right) \right). \end{aligned}$$

In general, if $A, B \subseteq X$, then $K \subseteq A \cup (X \setminus B)$ if and only if $K \cap B \subseteq A$. So

$$(59) \quad K \cup \{\infty\} \subseteq C_j \Leftrightarrow \Psi(j) \neq \emptyset \text{ and } K \cap \left(\bigcap_{i \in \Psi(j)} \hat{I}_{\frac{i-1}{2}} \right) \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}}.$$

It is easy to conclude that the function $\Psi': \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$ defined by

$$\Psi'(j) = \left\{ \frac{i-1}{2} \mid i \in \Psi(j) \right\} \text{ if } \Psi(j) \neq \emptyset, \text{ and } \Psi'(j) = \{0\} \text{ if } \Psi(j) = \emptyset$$

is c.f.v. Since the set $\{(j, l) \in \mathbb{N}^2 \mid \Psi'(j) = [l]\}$ is computable (Proposition 2.3(3)) and for each $j \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $\Psi'(j) = [l]$, there exists a computable function $g: \mathbb{N} \rightarrow \mathbb{N}$ such that $\Psi'(j) = [g(j)]$ for each $j \in \mathbb{N}$.

Let $j \in \mathbb{N}$ be such that $\Psi(j) \neq \emptyset$. Then

$$\bigcap_{i \in \Psi(j)} \hat{I}_{\frac{i-1}{2}} = \bigcap_{i \in \Psi'(j)} \hat{I}_i = \bigcap_{i \in [g(j)]} \hat{I}_i = L_{g(j)}.$$

By (59) for each $j \in \mathbb{N}$ we have

$$(60) \quad K \cup \{\infty\} \subseteq C_j \Leftrightarrow \Psi(j) \neq \emptyset \text{ and } K \cap L_{g(j)} \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}}.$$

The metric space (X, d) is unbounded by the assumption of the proposition. It is easy to conclude that there exists a computable function $\gamma: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$I_i \diamond I_{\gamma(i)}$$

for each $i \in \mathbb{N}$.

As above, we conclude that there exists a computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(61) \quad \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}} = J_{f(j)}$$

for each $j \in \mathbb{N}$ such that $\Phi(j) \neq \emptyset$ and

$$J_{f(j)} = I_{\gamma((g(j))_0)}$$

for each $j \in \mathbb{N}$ such that $\Phi(j) \neq \emptyset$. In the second case, since $I_{\gamma((g(j))_0)} \diamond I_{(g(j))_0}$, we have $I_{\gamma((g(j))_0)} \cap \hat{I}_{(g(j))_0} = \emptyset$ and consequently

$$(62) \quad J_{f(j)} \cap L_{g(j)} = \emptyset.$$

We claim that for each $j \in \mathbb{N}$ the following equivalence holds:

$$(63) \quad K \cup \{\infty\} \subseteq C_j \Leftrightarrow \Psi(j) \neq \emptyset \text{ and } K \cap L_{g(j)} \subseteq J_{f(j)}.$$

Suppose $j \in \mathbb{N}$ is such that $K \cup \{\infty\} \subseteq C_j$. It follows from (60) that $\Psi(j) \neq \emptyset$ and $K \cap L_{g(j)} \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}}$.

If $\Phi(j) \neq \emptyset$, then, by (61), $K \cap L_{g(j)} \subseteq J_{f(j)}$. If $\Phi(j) = \emptyset$, then $K \cap L_{g(j)} = \emptyset$ and $K \cap L_{g(j)} \subseteq J_{f(j)}$. In either case we have

$$(64) \quad \Psi(j) \neq \emptyset \text{ and } K \cap L_{g(j)} \subseteq J_{f(j)}.$$

Conversely, suppose $j \in \mathbb{N}$ is such that (64) holds.

If $\Phi(j) \neq \emptyset$, then $K \cap L_{g(j)} \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}}$ and it follows from (60) that $K \cup \{\infty\} \subseteq C_j$.

If $\Phi(j) = \emptyset$, then, by (62), $L_{g(j)} \cap J_{f(j)} = \emptyset$ which, together with $K \cap L_{g(j)} \subseteq J_{f(j)}$, gives $K \cap L_{g(j)} = \emptyset$. So $K \cap L_{g(j)} \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}}$ and (60) implies $K \cup \{\infty\} \subseteq C_j$.

So (63) holds. It follows readily from Proposition 8.7 and (63) that the set (58) is c.e.

(i) Suppose K is compact in (X, d) . Since (X, \mathcal{T}_d) is a subspace of (Y, \mathcal{S}) , we have that K is compact in (Y, \mathcal{S}) . To prove that the set

$$(65) \quad \{j \in \mathbb{N} \mid K \subseteq C_j\}$$

is c.e., we proceed in a similar way as in (ii). First, for each $j \in \mathbb{N}$ we get

$$K \subseteq C_j \Leftrightarrow K \cap \left(\bigcap_{i \in \Psi(j)} \hat{I}_{\frac{i-1}{2}} \right) \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}},$$

where we take $\bigcap_{i \in \Psi(j)} \hat{I}_{\frac{i-1}{2}} = X$ if $\Psi(j) = \emptyset$. Since K is bounded in (X, d) , there exists $i_0 \in \mathbb{N}$ such that $K \subseteq \hat{I}_{i_0}$. Let us take a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\bigcap_{i \in \Psi(j)} \hat{I}_{\frac{i-1}{2}} = L_{g(j)}$$

for each $j \in \mathbb{N}$ such that $\Psi(j) \neq \emptyset$ and

$$L_{g(j)} = \hat{I}_{i_0}$$

for each $j \in \mathbb{N}$ such that $\Psi(j) = \emptyset$. Then, for each $j \in \mathbb{N}$,

$$K \subseteq C_j \Leftrightarrow K \cap L_{g(j)} \subseteq \bigcup_{i \in \Phi(j)} I_{\frac{i}{2}}.$$

Now, in the same way as in (ii), we get that the set (65) is c.e. Thus K is semi-computable in $(Y, \mathcal{S}, (B_i))$. \square

Proposition 8.9. *Let (X, d, α) be a computable metric space and let $(Y, \mathcal{S}, (B_i))$ be its pseudocompactification. Suppose $K \subseteq X$ is such that $K \cup \{\infty\}$ is a c.e. set in $(Y, \mathcal{S}, (B_i))$. Then K is c.e. in (X, d, α) .*

Proof. Since $K \cup \{\infty\}$ is closed in (Y, \mathcal{S}) , (X, \mathcal{T}_d) is a subspace of (Y, \mathcal{S}) and $K = (K \cup \{\infty\}) \cap X$, we have that K is closed in (X, d) . The set

$$\Gamma = \{i \in \mathbb{N} \mid B_i \cap (K \cup \{\infty\}) \neq \emptyset\}$$

is c.e. by the assumption of the proposition. Let $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(i) = 2i$. For each $i \in \mathbb{N}$ we have

$$I_i \cap K \neq \emptyset \Leftrightarrow B_{2i} \cap (K \cup \{\infty\}) \neq \emptyset \Leftrightarrow 2i \in \Gamma \Leftrightarrow f(i) \in \Gamma \Leftrightarrow i \in f^{-1}(\Gamma).$$

Thus $\{i \in \mathbb{N} \mid I_i \cap K \neq \emptyset\} = f^{-1}(\Gamma)$ and the claim follows. \square

As noted, the implication

$$(66) \quad \partial K \text{ computable} \implies K \text{ computable}$$

need not hold if K is a noncompact semicomputable manifold with boundary. We are going to prove that (66) holds in the special case when K is homeomorphic to \mathbb{R}^n or \mathbb{H}^n . Moreover, we will get that (66) holds if a sufficiently large part of K looks like \mathbb{R}^n or \mathbb{H}^n . More precisely, we will observe a manifold K for which there exists an open set $U \subseteq K$ such that \overline{U} is compact and such that $K \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0, r)$ or $\mathbb{H}^n \setminus B(0, r)$, where $r > 0$ and $B(0, r)$ is an open ball in \mathbb{R}^n with respect to the Euclidean metric. We may assume $r = 1$ since $\mathbb{R}^n \setminus B(0, r) \cong \mathbb{R}^n \setminus B(0, 1)$ and $\mathbb{H}^n \setminus B(0, r) \cong \mathbb{H}^n \setminus B(0, 1)$ (we use $X \cong Y$ to denote that topological spaces X and Y are homeomorphic). Furthermore, it is not hard to conclude that $\mathbb{H}^n \setminus B(0, 1) \cong \mathbb{H}^n$.

For $n \geq 1$ let

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\},$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n .

If X is a topological space and $A \subseteq X$, by \overline{A} we denote the closure of A in X .

Lemma 8.10. *Let $n \geq 1$ and let K be an n -manifold with boundary. Suppose that there exists an open set $U \subseteq K$ such that \overline{U} is compact and $K \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0, 1)$ or \mathbb{H}^n . Then the following holds.*

- (i) *A one-point compactification $K \cup \{\infty\}$ of K is an n -manifold with boundary; if $K \setminus U \cong \mathbb{R}^n \setminus B(0, 1)$, the boundary of $K \cup \{\infty\}$ is ∂K , and if $K \setminus U \cong \mathbb{H}^n$, the boundary of $K \cup \{\infty\}$ is $\partial K \cup \{\infty\}$.*
- (ii) *If $K \setminus U \cong \mathbb{R}^n \setminus B(0, 1)$, then ∂K is compact. If $K \setminus U \cong \mathbb{H}^n$, then ∂K is not compact.*

Proof. (i) In general, if $X \cup \{\infty\}$ is a one-point compactification of a topological space X , then X is clearly an open subspace of $X \cup \{\infty\}$. Therefore, if $x \in K$ and N a neighborhood of x in K , then N is a neighborhood of x in $K \cup \{\infty\}$. This means that we only have to prove that ∞ has a neighborhood in $K \cup \{\infty\}$ which is homeomorphic either to \mathbb{R}^n (if $K \setminus U \cong \mathbb{R}^n \setminus B(0, 1)$) or \mathbb{H}^n by a homeomorphism which maps ∞ to $\text{Bd } \mathbb{H}^n$ (if $K \setminus U \cong \mathbb{H}^n$).

Since \overline{U} is compact, the set $(K \setminus \overline{U}) \cup \{\infty\}$ is open in $K \cup \{\infty\}$ and obviously $(K \setminus \overline{U}) \cup \{\infty\} \subseteq (K \setminus U) \cup \{\infty\}$. So $(K \setminus U) \cup \{\infty\}$ is a neighborhood of ∞ in $K \cup \{\infty\}$.

It is easy to verify the following general fact: if Y is a closed subspace of a topological space X and $Y \cup \{\infty\}$ and $X \cup \{\infty\}$ are one-point compactifications of Y and X , then $Y \cup \{\infty\}$ is a subspace of $X \cup \{\infty\}$.

Therefore, the compactification $(K \setminus U) \cup \{\infty\}$ of $K \setminus U$ is a subspace of $K \cup \{\infty\}$. Using the fact that $(K \setminus U) \cup \{\infty\}$ is a neighborhood of ∞ in $K \cup \{\infty\}$, we conclude the following: if N is a neighborhood of ∞ in $(K \setminus U) \cup \{\infty\}$, then N is also a neighborhood of ∞ in $K \cup \{\infty\}$. So it suffices to find a neighborhood of ∞ in $(K \setminus U) \cup \{\infty\}$ with desired properties.

Let us suppose that $K \setminus U \cong \mathbb{R}^n \setminus B(0, 1)$. It suffices to prove that the point ∞ in the one-point compactification $(\mathbb{R}^n \setminus B(0, 1)) \cup \{\infty\}$ has a neighborhood homeomorphic to \mathbb{R}^n . But, as above, $(\mathbb{R}^n \setminus B(0, 1)) \cup \{\infty\}$ is a subspace of $\mathbb{R}^n \cup \{\infty\}$ and $(\mathbb{R}^n \setminus B(0, 1)) \cup \{\infty\}$ is a neighborhood of ∞ in $\mathbb{R}^n \cup \{\infty\}$. So it is enough to prove that ∞ has a neighborhood in $\mathbb{R}^n \cup \{\infty\}$ which is homeomorphic to

\mathbb{R}^n . However, this is clear since $\mathbb{R}^n \cup \{\infty\}$ is homeomorphic to \mathbb{S}^n and \mathbb{S}^n is an n -manifold.

Let us suppose now that $K \setminus U \cong \mathbb{H}^n$. Since $\mathbb{H}^n \cong \mathbb{H}^n \cap B(0, 1)$ by the homeomorphism $x \mapsto \frac{x}{1+\|x\|}$, we have

$$K \setminus U \cong B(0, 1) \cap \mathbb{H}^n.$$

So it is enough to prove that ∞ has a neighborhood in $(B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\}$ which is homeomorphic to \mathbb{H}^n by a homeomorphism which maps ∞ to $\text{Bd } \mathbb{H}^n$.

Since the set $B(0, 1) \cap \mathbb{H}^n$ is closed in $B(0, 1)$, $(B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\}$ is a subspace of $B(0, 1) \cup \{\infty\}$. It is known that the function $f : B(0, 1) \cup \{\infty\} \rightarrow \mathbb{S}^n$ given by $f(\infty) = (-1, 0, \dots, 0)$, $f(0) = (1, 0, \dots, 0)$ and

$$f(x) = \left(\cos \|x\|, \frac{x_1}{\|x\|} \sin \|x\|, \dots, \frac{x_n}{\|x\|} \sin \|x\| \right)$$

for $x \in B(0, 1)$, $x \neq 0$, $x = (x_1, \dots, x_n)$, is a homeomorphism. This function induces a homeomorphism

$$(B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\} \rightarrow f((B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\}).$$

However,

$$f((B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\}) = \mathbb{S}^n \cap \mathbb{H}^{n+1}$$

and $\mathbb{S}^n \cap \mathbb{H}^{n+1}$, i.e. upper half-sphere, is an n -manifold with boundary, its boundary is $\mathbb{S}^{n-1} \times \{0\}$. We conclude that $(B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\}$ is an n -manifold with boundary and ∞ belongs to its boundary, meaning that ∞ has a desired neighborhood in $(B(0, 1) \cap \mathbb{H}^n) \cup \{\infty\}$.

(ii) Suppose $f : \mathbb{R}^n \setminus B(0, 1) \rightarrow K \setminus U$ is a homeomorphism. Then $K \setminus U$ is Hausdorff and since the set $f(\mathbb{S}^{n-1})$ is compact in $K \setminus U$, this set is closed in $K \setminus U$. But $K \setminus U$ is closed in K , so $f(\mathbb{S}^{n-1})$ is closed in K . It follows that the set

$$A = K \setminus (\overline{U} \cup f(\mathbb{S}^{n-1}))$$

is open in K . We have $A \subseteq K \setminus U$, so A is open in $K \setminus U$ and it is therefore homeomorphic to the open subset $f(A)$ of $\mathbb{R}^n \setminus B(0, 1)$. It follows from the definition of A that $f(A) \subseteq \mathbb{R}^n \setminus \hat{B}(0, 1)$ and, since the set $\mathbb{R}^n \setminus \hat{B}(0, 1)$ is open in \mathbb{R}^n , $f(A)$ is open in \mathbb{R}^n . Hence A is open subset of K which is homeomorphic to an open subset of \mathbb{R}^n and it follows that each point of A has a neighborhood in K homeomorphic to \mathbb{R}^n . So $A \cap \partial K = \emptyset$, hence $\partial K \subseteq \overline{U} \cup f(\mathbb{S}^{n-1})$. In general, the boundary of a manifold is a closed subset of the manifold. As a closed set contained in a compact set, ∂K is compact.

Suppose $f : \mathbb{H}^n \rightarrow K \setminus U$ is a homeomorphism. Since $(K \setminus U) \cap \overline{U}$ is compact, the preimage by f of this set is compact in \mathbb{H}^n and we conclude that there exists $r > 0$ such that $f(\mathbb{H}^n \setminus \hat{B}(0, r)) \subseteq K \setminus \overline{U}$. It follows that the set $f(\mathbb{H}^n \setminus \hat{B}(0, r))$ is open in K and, consequently,

$$(67) \quad f(\text{Bd } \mathbb{H}^n \setminus \hat{B}(0, r)) \subseteq \partial K.$$

Suppose ∂K is compact. The set $(K \setminus U) \cap \partial K$ is closed and contained in ∂K , hence it is compact. So $f^{-1}(\partial K)$ is a compact set in \mathbb{H}^n . But this contradicts (67). Thus ∂K is not compact. \square

Theorem 8.11. *Let (X, d, α) be a computable metric space and let K be a semi-computable set in this space which is, as a subspace of (X, d) , a manifold with boundary. Then the implication*

$$\partial K \text{ computable} \implies K \text{ computable}$$

holds if there exists an open set U in K such that \overline{U} is compact in K and $K \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0, 1)$ or \mathbb{H}^n .

Proof. We may assume that K is noncompact, otherwise the claim follows from Theorem 7.3 (or [10]). It follows that (X, d) is unbounded.

Suppose that ∂K is semicomputable in (X, d, α) . Let $(Y, \mathcal{S}, (B_i))$ be a pseudo-compactification of (X, d, α) .

Let us suppose that $K \setminus U \cong \mathbb{R}^n \setminus B(0, 1)$ for some open set U in K such that \overline{U} is compact. By Lemma 8.10(ii) the set ∂K is compact and, by Proposition 8.8(i), ∂K is semicomputable in $(Y, \mathcal{S}, (B_i))$. By Proposition 8.8(ii), $K \cup \{\infty\}$ is semicomputable in $(Y, \mathcal{S}, (B_i))$ and, by Lemma 8.10(i) and Proposition 8.4, $K \cup \{\infty\}$ is a manifold with boundary and its boundary is ∂K . It follows from Theorem 7.3 that $K \cup \{\infty\}$ is c.e. in $(Y, \mathcal{S}, (B_i))$. Proposition 8.9 now implies that K is c.e. in (X, d, α) . Hence K is computable in (X, d, α) .

Let us suppose that $K \setminus U \cong \mathbb{H}^n$ for some open set U in K such that \overline{U} is compact. Using Proposition 8.4, Lemma 8.10 and Proposition 8.8 we get the following conclusion: $K \cup \{\infty\}$ is a semicomputable manifold with boundary in $(Y, \mathcal{S}, (B_i))$, its boundary is $\partial K \cup \{\infty\}$ and $\partial K \cup \{\infty\}$ is a semicomputable set in $(Y, \mathcal{S}, (B_i))$. Again, Theorem 7.3 and Proposition 8.9 imply that K is computable in (X, d, α) . \square

In particular, if K is a semicomputable set in (X, d, α) such that $K \cong \mathbb{R}^n$, then K is computable, and if K is a semicomputable set for which there exists a homeomorphism $f : \mathbb{H}^n \rightarrow K$ such that $f(\text{Bd } \mathbb{H}^n)$ is a semicomputable set, then K is computable.

Example 8.12. Let (X, d, α) be a computable metric space and let K be a semi-computable set in this space.

Suppose $K \cong \mathbb{S}^1 \times [0, \infty)$. Then K is a manifold with boundary and $\partial K = f(\mathbb{S}^1 \times \{0\})$, where $f : \mathbb{S}^1 \times [0, \infty) \rightarrow K$ is a homeomorphism. Suppose ∂K is semicomputable set. Then K is computable. This follows from Theorem 8.11 since $\mathbb{S}^1 \times [0, \infty) \cong \mathbb{R}^2 \setminus B(0, 1)$: the function $g : \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{R}^2 \setminus B(0, 1)$, $g(x, t) = (1 + t)x$, is a homeomorphism.

If we restrict g to the product of the upper half-circle $\mathbb{S}^1 \cap \mathbb{H}^2$ and $[0, \infty)$, we get the conclusion that $[0, 1] \times [0, \infty) \cong \mathbb{H}^2 \setminus B(0, 1)$, hence $[0, 1] \times [0, \infty) \cong \mathbb{H}^2$.

Suppose $K \cong [0, 1] \times [0, \infty)$. Then K is a manifold with boundary and $\partial K = f([0, 1] \times \{0\} \cup \{0, 1\} \times [0, \infty))$, where $f : [0, 1] \times [0, \infty) \rightarrow K$ is a homeomorphism. By Theorem 8.11, K is computable if ∂K is semicomputable.

9. CONCLUSION

Semicomputable sets in Euclidean spaces (and in other usual spaces) naturally arise and it is of interest to know under which conditions these sets are computable. It is known that topology plays a important role in the description of such conditions. In particular, a semicomputable set is computable if it is a compact topological manifold (whose boundary is semicomputable). In this paper we have shown that topology is actually involved in this matter at the basic level: the ambient

space (Euclidean space or computable metric space) can be replaced by a computable topological space. Hence, to define necessary notions and to prove that semicomputable sets are computable under certain conditions, we do not need Euclidean space and we do not need even metric spaces: computable topological spaces are sufficient.

Furthermore, it has been shown how the introduced concepts and results can be used to conclude that certain noncompact sets in computable metric spaces are computable. We believe that the subject of this paper has a potential for further investigations and applications.

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